O.O. Sdvyzhkova
Y.B. Olevska
D.V. Babets
L.I. Korotka

ORDINARY DIFFERENTIAL EQUATIONS

STUDY GUIDE

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ORDINARY DIFFERENTIAL EQUATIONS

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Рецензенти:

О.Б. Іванов – канд. техн. наук, проф. Державного вищого навчального закладу «Національний гірничий університет», проф. кафедри відновлюваних джерел енергії;

Т.С. Кагадій – д-р фіз.-мат. наук, проф. Державного вищого навчального закладу «Національний гірничий університет», проф. кафедри вищої математики.

Ordinary differential equations = Звичайні диференційні рівняння:
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This study guide provides the basic concepts and definitions of differential equation theory. It highlights the most important integration methods and theorems of solution existence.

This textbook should be used for educational purpose by university students studying technical science.

Наведено основні поняття та визначення теорії звичайних диференціальних рівнянь. Висвітлено найбільш важливі методи інтегрування та теореми існування розв'язків.

Для студентів вищих навчальних закладів технічних спеціальностей різних форм навчання.

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1. CONCEPT OF DIFFERENTIAL EQUATION AND ITS SOLUTION

Differential equations arise in the study of various problems in science, engineering, economics. In the language of differential equations are formulated the most important laws of nature. In the process of solving various problems is not always possible to establish a direct relationship between the quantities that describe the process. However, in most cases we can establish a relationship between quantities and their velocities of change relative to other variable quantities. This means that will be found the equations in which unknowns are included under the sign of the derivative. These equations are called by **differential equations**.

Definition 1.1. Equations in which the unknown function appears under the sign of the derivative or the differential are called **differential equations** (**DE**).

There is one differential equation which is probably known by everyone. It is the Newton's Second Law of Motion. If an object with a mass m is moving with acceleration a under the force F then Newton's Second Law tells us

$$F = ma\,.$$

To see that this it is really a differential equation we have to rewrite it a little. First, remember that we can rewrite the acceleration *a*, by one of two ways:

$$a = \frac{dv}{dt} \quad \text{or} \quad a = \frac{d^2u}{dt^2},$$

where v is a velocity of an object and u is a position function of an object at any time *t*. At this point we should also remember that the force F may also be a function of time, velocity and position.

So, with all these things in mind the Newton's Second Law can now be written as a differential equation in terms of either a velocity v or a position of an object u as follows:

$$m\frac{dv}{dt} = F(t,v),$$

$$m\frac{d^2u}{dt^2} = F\left(t,v,\frac{du}{dt}\right).$$

So, here is our first differential equation. We will see both forms of this in later chapters. Here are a few more examples of differential equations:

1) $y' = x^2 + 2xy$; 2) $\frac{\partial^2 y}{\partial x^2} + 2y + x = 0$;

3) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$; 4) $x^2dy + ydx = 0$.

Definition 1.2. An ordinary differential equation is a differential equation in which an unknown function (also known as a dependent variable) is a function of a single independent variable.

Definition 1.3. A partial differential equation is a differential equation in which an unknown function is a function of multiple independent variables and the equation involves its partial derivatives. For example, the equations 1, 2, 4 are ordinary differential equations and equation 3 is a partial differential equation.

The highest derivative that appears in a differential equation gives an order. Thus, 1 and 4 – are the first order differential equations, and 2 and 3 ones are the second order differential equations.

We will consider only ordinary differential equations.

General view of an ordinary n-th order differential equation is:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{1.1}$$

where x is independent variable; y is unknown function of x which is to be determined; $y', \ldots, y^{(n)}$ are derivatives of function, F is the given function of $(n+2)$ variables that changes in some area $\Im$. The area $\Im$ is called as a default domain of the equations.

Definition 1.4. Solution to a differential equation is any differentiable function $y = \varphi(x)$ which satisfies the differential equation together with its derivatives.

For example let us consider the equation $y'' + y = 0$. It can be shown easily that function $y = \sin x$ satisfies this equation. So it is a solution to the equation. But following functions $y = 2\cos x$, $y = 3\sin x - \cos x$ and in general $y = C_1 \cos x + C_2 \sin x$ (where C_1, C_2 are arbitrary constants) are the solutions to given equation as well. This is easily seen if we substitute the function in equation. So in fact there are an infinite number of solutions of this differential equation.

Definition 1.5. To solve a differential equation means to find all its solutions.

A general solution of a differential equation is the most general form that the solution can take. This solution is given in terms of n independent parameters, in other words it contains arbitrary constants $C_1, C_2, \ldots, C_n$ where n is an order of a differential equation.

Hence, a function $y = \varphi(x, C)$ is a general solution to the differential equation of the first order $F(x, y, y') = 0$.

Respectively a function $y = \varphi(x, C_1, C_2)$ is a general solution to the differential equation of the second order $F(x, y, y', y'') = 0$.

And finally a function $y = \varphi(x, C_1, \ldots, C_n)$ is a general solution to the n- order differential equation $F\left(x, y, y', \ldots, y^{(n)}\right) = 0$.

Definition 1.6. A particular solution of a differential equation (relative to a general solution) is a solution which could be obtained from that general solution by simply choosing specific values of the parameters involved. To determine these parameters we need some additional conditions which are, for example, some values of a unknown function and its derivative(s) at specific points. The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

Graph of the particular solution is called an integral curve of equation. Therefore, in terms of the geometry a general solution which includes all particular solutions is a set of integral curves.

2. CAUCHY'S PROBLEM FOR THE FIRST ORDER DIFFERENTIAL EQUATION

Let us consider the first order differential equation solved for the first derivative

$$y' = f(x, y), \tag{2.1}$$

where $f(x, y)$ is a function given at domain $\Im$ on the plane xOy.

As we noted above in the Chapter 1 there are an infinite number of solutions to this differential equation. But actually in practical problems we need not all solution, i.e. not a general solution, but only that one satisfied some additional conditions.

So called Cauchy's problem is one of such problem. It is formulated for the differential equation (2.1) as following: to find among all solution of the differential equation (2.1) the function $y = y(x)$ which gives the value y_0 when the argument has given value $x = x_0$, i.e.

$$y(x_0) = y_0. \tag{2.2}$$

The values x_0, y_0 are called initial date and the conditions (2.2) itself is called initial conditions. Sometimes the solution cannot be obtained in explicit form as a function waged relatively y. So an explicit solution is any solution that is given in the form $y = \varphi(x)$. An implicit solution is any solution that isn't in explicit form.

Cauchy's problem for the differential equation (2.1) is not always solvable. Cauchy's problem has no solution if there is no function $y = y(x)$ that would satisfy the differential equation (2.1) on the interval (a, b) and the initial condition (2.2). Cauchy problem has a number solutions if there are several functions that satisfy (2.1) and (2.2).

Theorem 2.1 (Cauchy's theorem on the existence and unity of a solution). Let a function $f(x, y)$ is defined in some region $\Im$ of the plane xOy. Let a point $(x_0, y_0) \in \Im$ and next conditions are satisfied for the function $f(x, y)$ in $\Im$:

1) $f(x, y)$ is a continuous function of two variables x, y;

2) $f(x, y)$ has continuous partial derivatives $\dfrac{\partial f}{\partial y}$.

Then there is an interval such that $(x_0 - h, x_0 + h)$ on the axis Ox where a unique solution $y = \varphi(x)$ of (2.1) exist and $\varphi(x_0) = y_0$.

Let the Cauchy's problem for the equation (2.1) in $\Im$ has a unique solution. Then only one integral curve passes through each point of the region (fig. 2.1).

Example 2.1.

$$y' = xy + e^{-y}.$$

The right side of this equation $f(x, y) = xy + e^{-y}$ and its partial derivatives $\frac{\partial f}{\partial y} = x - e^{-y}$ are continuous at all points of the plane xOy. According to the Theorem 2.1 there is one integral curve of the original equation which passes through each point (x_0, y_0) of the plane xOy.

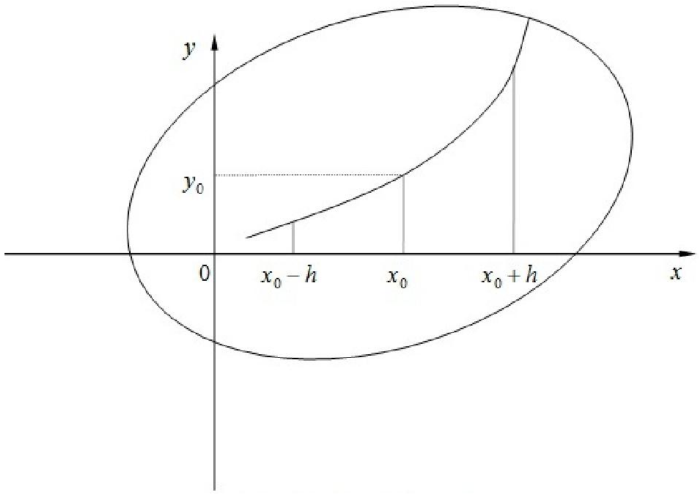

Fig. 2.1. Cauchy's problem

Example 2.2.

$$y' = 3y^{2/3}. \tag{2.3}$$

The function $f(x, y) = 3y^{2/3}$ is continuous on the whole plane xOy. The partial derivative $\frac{\partial f}{\partial y} = \frac{2}{\sqrt[3]{y}}$ is continuous at all points where $y \neq 0$. Therefore, the uniqueness may be violated at the points of axis Ox $(y = 0)$. So the function

$$y = (x - C)^3 \tag{2.4}$$

is a solution of the equation (2.3), C is an integration constant. Then at least two integral curves pass through each point of the axis Ox. They are an integral curve of the curve set (2.4) and the axis Ox itself. The uniqueness of the solution is violated at the points on the axis Ox (fig. 2.2).

Both conditions of Cauchy's theorem are satisfied in the region where $y \neq 0$:

$$D = \{-\infty < x < +\infty, y \neq 0\}.$$

Thus D is the region of equation (2.3) unity. Only one integral curve passes through each point $(x_0, y_0) \in D$.

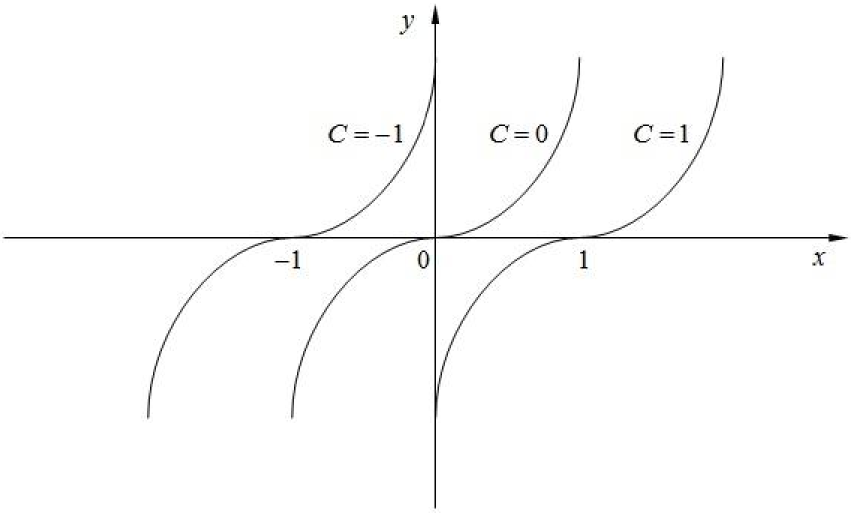

Fig. 2.2. Uniqueness of the solution

Consider the n-th order differential equation which resolved with respect to the highest derivatives

$$y^{(n)} = f\left(x, y, y', \ldots, y^{(n-1)}\right). \tag{2.5}$$

The Cauchy's problem means to find a solution $y = y(x)$ among all solutions of equation (2.5) for which

$$y(x_0) = y_0,\ y'(x_0) = y'_0, \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)}, \tag{2.6}$$

where y_0, $y'_0, \ldots,$ $y_0^{(n-1)}$ are arbitrary real numbers. The numbers x_0, y_0, $y'_0, \ldots,$ $y_0^{(n-1)}$ are called the **initial data** of the equation (2.5).

For second-order differential equation

$$y'' = f(x, y, y'). \tag{2.7}$$

The Cauchy problem means finding solutions $y = y(x)$ which satisfies the initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y'_0. \tag{2.8}$$

Consider the mechanical interpretation of Cauchy's problem for second order.

Let a body M moves rectilinearly along the axis Ox. Then at a moment of time t a value x gives the body position. Then $\frac{dx}{dt}$ is the velocity of this body t and

$\frac{d^2x}{dt^2}$ is the acceleration. Let the force $f\left(t, x, \frac{dx}{dt}\right)$ acting upon the body M dependents on such variables as time, position and velocity. Let the mass of the body is equal to 1. Then, according to the Newton's second law we obtain a second-order differential equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right). \tag{2.9}$$

It describes the displacement of the body along a line. Any solution of this equation corresponds to some possible displacement. The Cauchy's problem for the equation (2.9) means that among all possible displacement defined by the equation (2.9), we need to find a displacement $x = x(t)$ that satisfies the initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = \left.\frac{dx}{dt}\right|_{t=t_0} = v_0. \tag{2.10}$$

It means to find such a displacement in which the moving body would have a position x_0 and initial velocity v_0 at time t_0.

Let us consider the n-th order equation.

Theorem 2.2. Suppose we have the equation (2.5) where the function $f\left(x, y, y', \ldots, y^{(n-1)}\right)$ is defined in some region $\Im$ of variables $x, y, y', \ldots, y^{(n-1)}$ which contains the point $M_0\left(x_0, y_0, y_0', \ldots, y_0^{(n-1)}\right)$. For the function f in a region $\Im$ such conditions are performed:

1) $f\left(x, y, y', \ldots, y^{(n-1)}\right)$ is a continuous function of its arguments;

2) $f\left(x, y, y', \ldots, y^{(n-1)}\right)$ has continuous partial derivatives

$$\frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'}, \ldots, \frac{\partial f}{\partial y^{(n-1)}}.$$

Then there is an interval $(x_0 - h, x_0 + h)$ of the axis Ox on which there is a unique solution $y = \varphi(x)$ of the equation (2.5) satisfying the initial conditions (2.6).

Consider the equation

$$y'' = \sin y' + e^{-x^2} y \tag{2.11}$$

and initial conditions

$$y(x_0) = y_0, \; y'(x_0) = y_0'. \tag{2.12}$$

In this case $f(x,y,y')=\sin y'+e^{-x^2}y$, $\dfrac{\partial f}{\partial y}=e^{-x^2}$, $\dfrac{\partial f}{\partial y'}=\cos y'$. The function f and its partial derivatives $\dfrac{\partial f}{\partial y}$, $\dfrac{\partial f}{\partial y'}$ are the continuous functions for all values of its variables x,y,y'. According to the Theorem 2.2 for any numbers x_0,y_0,y_0' there is only one solution of the equation (2.11) that satisfies the initial conditions (2.12).

3. GEOMETRIC INTERPRETATION OF FIRST ORDER DIFFERENTIAL EQUATIONS

Differential equation

$$y'=f(x,y) \tag{3.1}$$

can be given a geometric interpretation. Let $\Im$ is the domain of definition for the equation (3.1). Let the graph of the solution $y=\varphi(x)$ of the equation (3.1) passes through the point $(x_0,y_0)\in\Im$, i.s. $\varphi(x_0)=y_0$. Tangent line equation at this point looks like

$$y=f(x_0,y_0)(x-x_0)+y_0 \tag{3.2}$$

because $\varphi'(x_0)=f(x_0,\varphi(x_0))=f(x_0,y_0)$. Hence, the differential equation (3.1) puts a straight line (3.2) with a slope $f(x_0,y_0)$ to each point $(x_0.y_0)\in\Im$.

Therefore, the equation (3.1) gives the directions of tangent lines for integral curves. This interpretation shows a way to solve the equation graphically.

4. FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

Consider the individual types of first order differential equations.

A first order differential equation which is solved respectively the derivative looks like

$$y'=f(x,y), \qquad x'=g(x,y) \tag{4.1}$$

or

$$M(x,y)dx+N(x,y)dy=0, \tag{4.2}$$

where f, g, M, N are known functions. Variables x and y are included equally in the form (4.2). It does not matter which variable is a function and which variable is an argument.

Consider the concept of the equation with ***separated*** variables. The equation (4.2) in which the coefficient of dx is a function depending only on x and the coefficient of dy is a function depending only on y, such that

$$F_1(x)dx+F_2(y)dy=0 \tag{4.3}$$

is called the differential equation with separated variables.

Assume that a solution of the equation (4.3) is a function $y=\varphi(x)$. We prove an identity by substituting it into the equation (4.3):

$$F_1(x)dx+F_2\left[\varphi(x)\right]\varphi'(x)dx=0.$$

This identity is an identity of differentials. One side of the identity is expressed directly through an independent variable, and another one contains a function. If differentials are equal then their integrals can differ only with a constant. So

$$\int F_1(x)dx+\int F_2(y)dy=C \tag{4.4}$$

is a general integral of the equation (4.3).

For example $xdx+e^y dy=0$ is an equation with separated variables. We obtain

$$\int xdx+\int e^y dy=C, \quad \frac{x^2}{2}+e^y=C.$$

Consider the concept of the equation with ***separable*** variables. An equation of the form

$$y'=f_1(x)f_2(y) \tag{4.5}$$

or

$$\varphi_1(x)\varphi_2(y)dx+\psi_1(x)\psi_2(y)dy=0 \tag{4.6}$$

is called the differential equation with separable variables. The right side of the equation (4.5) and each of the differential coefficients in the equation (4.6) are a product of two functions. One of these functions depends only on x and the second one depends only on y.

To solve on the equation (4.5) or (4.6) we transform it the form (4.3) in which a coefficient of dx depends only on x and a coefficient of dy depends only on y. Then divide both part of equation by the product $\varphi_2(y)\psi_1(x)$. Here it is assumed that $\psi_1(x)\neq 0$ and $\varphi_2(y)\neq 0$. After dividing one can obtain

$$\frac{\varphi_1(x)}{\psi_1(x)}dx+\frac{\psi_2(y)}{\varphi_2(y)}dy=0.$$

This equation is an equation with separated variables. Integrating one can obtain

$$\int\frac{\varphi_1(x)}{\psi_1(x)}dx+\int\frac{\psi_2(y)}{\varphi_2(y)}dy=C. \tag{4.7}$$

This is a general integral of the equation (4.6). We should also separately explore the values x and y at which the functions $\varphi_2(y)$ and $\psi_1(x)$ are zero

$$\varphi_2(y)=0, \quad \psi_1(x)=0.$$

Roots of these equations $y=a=const$ and $x=b=const$ are the solutions of the equation (4.6) as well because of equalities:

$$\varphi_2(a)=0, \quad da=0, \quad \psi_1(b)=0, \quad db=0.$$

These solutions do not necessarily belong to the set of solutions defined by (4.7).

To separate the variables in the equation (4.5) it should be divided by $f_2(y)$ and multiplied by dx:

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C.$$

Here $y=y_0$ is a solution of the equation (4.5) if $f_2(y_0)=0$.

Example 4.1. Integrate the equation

$$x(y^2-1)dx + y(x^2-1)dy = 0. \tag{4.8}$$

Solution. The coefficient at dx $x(y^2-1)$ is the product of the function $\varphi_1(x)=x$ which depends only on x and the function $\varphi_2(y)=y^2-1$ which depends only on y. Similarly the coefficient at dy $y(x^2-1)$ is the product of the function $\psi_2(y)=y$ which depends only on y and the function $\psi_1(x)=x^2-1$ which depends only on x. So the equation (4.8) with separable variables of the form (4.6). Multiply both sides of this equation by the function $1/(x^2-1)(y^2-1)$. We obtain an equation with separated variables

$$\frac{xdx}{x^2-1} + \frac{ydy}{y^2-1} = 0, \quad (x^2-1)(y^2-1) \neq 0.$$

By integrating we have

$$\int \frac{xdx}{x^2-1} + \int \frac{ydy}{y^2-1} = C_1. \tag{4.9}$$

Here C_1 is an integral constant. Let us compute the integral $\int \frac{xdx}{x^2-1}$. We make a substitution $x^2-1=t$. Hence

$$\int \frac{xdx}{x^2-1} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| = \frac{1}{2}\ln\left|x^2-1\right|.$$

We put the result into the equality (4.9). Then

$$\frac{1}{2}\ln\left|x^2-1\right| + \frac{1}{2}\ln\left|y^2-1\right| = C_1,$$
$$\left(x^2-1\right)\left(y^2-1\right) = \pm e^{2C_1} = C_2, \quad C_2 \neq 0.$$

Considering $\left(x^2-1\right)\left(y^2-1\right) \neq 0$ we obtain a general integral

$$\left(x^2-1\right)\left(y^2-1\right) = C_2, \quad C_2 \neq 0. \tag{4.10}$$

We should find the roots of equations

$$x^2-1=0, \quad y^2-1=0.$$

Hence $x=\pm 1$, $y=\pm 1$ are particular solutions of (4.8). They are found from the general solution (4.10) at $C_2=0$. So it is not necessary to write them separately. Hence a general integral of the differential equation (4.8) is

$$\left(x^2-1\right)\left(y^2-1\right) = C,$$

where C is an integral constant.

Example 4.2. Find a general integral of the equation

$$y' = 3\sqrt[3]{y^2}. \tag{4.11}$$

Solution. This is an equation with separable variables of the form (4.5) where $f_1(x)=3$, $f_2(y)=\sqrt[3]{y^2}$.

Rewrite the equation as $\frac{dy}{dx} = 3\sqrt[3]{y^2}$. To separate the variables we multiply this equation by $dx/3\sqrt[3]{y^2}$. We obtain

$$\frac{dy}{3\sqrt[3]{y^2}} = dx, \quad y \neq 0.$$

Integrating we find

$$\int \frac{dy}{3\sqrt[3]{y^2}} = \int dx + C, \quad y^{1/3} = x + C,$$

$$y = (x + C)^3. \tag{4.12}$$

We should verify that a solution $y = 0$ is not lost (we divided both part of equation by $3\sqrt[3]{y^2}$). The solution $y = 0$ cannot be obtained from the general solution (4.12) for any value of the integral constant C. Therefore a general solution of this equation is written as follows:

$$y = (x + C)^3, \quad y = 0.$$

Example 4.3. A physical body with temperature at the initial time is T_0 is placed in an environment with a constant temperature θ. How the physical body's temperature vary with time?

Solution. Denote by $T(t)$ the physical body temperature at the moment of time t. Note that $T(0) = T_0$. Known fact is that the rate of a body temperature changes proportionally to the difference between the body temperature at a given time and environment temperature. This means that

$$T'(t) = -\gamma(T(t) - \theta), \tag{4.13}$$

where $\gamma > 0$ is a proportionality factor. The minus sign on the right side of the equation corresponds to the experimental data. If $T(t) - \theta > 0$ then a body's temperature decreases and therefore the rate of change is negative, and if $T(t) - \theta < 0$ then body temperature increases, and therefore the rate of change is positive.

Thus a process of heating (cooling) of a body is modeled by the equation (4.13).

The equation (4.13) is a one with separable variables. Let us solve it:

$$\frac{dT}{T - \theta} = -\gamma dt, \quad \int \frac{dT}{T - \theta} = -\int \gamma dt + C_1,$$

$$\ln|T - \theta| = -\gamma t + C_1, \quad |T - \theta| = e^{C_1 - \gamma t},$$

$$T - \theta = \pm e^{C_1 - \gamma t} = Ce^{-\gamma t} \quad \left(C = \pm e^{C_1}\right),$$

$$T(t) = \theta + Ce^{-\gamma t}. \tag{4.14}$$

The formula (4.14) includes all solutions of the equation (4.13). Considering the initial condition $T(0) = T_0$ we obtain formula $T_0 = \theta + C$, hence $C = T_0 - \theta$. Substituting C into the formula (4.14) we obtain the desired time-temperature function

$$T(t) = \theta + (T_0 - \theta)e^{-\gamma t}.$$

The function $T(t)$ increases when $T(t) - \theta < 0$ (a physical body heats up) and decreases when $T(t) - \theta > 0$ (a physical body cools). In both cases its value tends to θ with increase of parameter t.

Example 4.4. Let a long horizontal excavation be driven in the homogeneous rock mass at a depth **H** from a surface. The cross section of excavation has a circular outline with R_0 radius. The internal distributed presser p_0 is applied to the excavation contour. The specific gravity of rocks is **γ.** The task is to determine the distribution of stresses around the excavation.

We suppose the stresses in the rock mass depends only on values H and **γ** and does not surpass rock strength, i.e. stress-strain state of rock mass is elastic. Assume that the initial stress field is hydrostatic (fig 4.1).

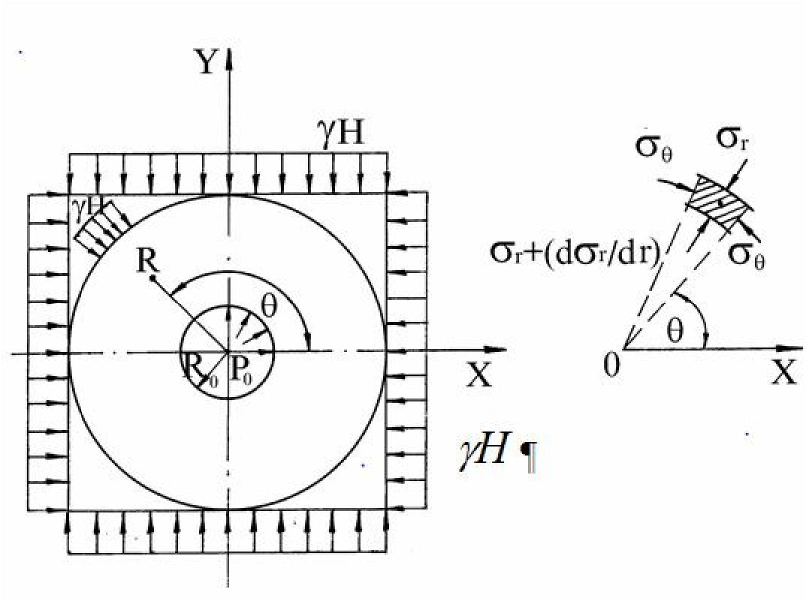

Fig. 4.1. The calculated scheme to determine the stress distribution around a long horizontal excavation

In this case this problem is polar-symmetric in which all geometric parameters and power elements depend only on a radius R and do not depend on a polar angle θ. We use an equilibrium condition:

$$\frac{d\sigma_r}{dr} - \frac{\sigma_\theta - \sigma_r}{r} = 0, \tag{4.15}$$

where σ_θ and σ_r are tangential and radial stress components, $r = \dfrac{R}{R_0}$ is a dimensionless radius. We use the condition of compatibility:

$$\sigma_\theta + \sigma_r = 2\gamma H\,. \tag{4.16}$$

The stress component σ_θ can be found from (4.16). Substituting this expression into (4.15) we obtain a differential equation

$$\frac{d\sigma_r}{dr} - 2\frac{\gamma H - \sigma_r}{r} = 0\,.$$

This is an equation with separable variables which after separation of variables takes the form

$$\frac{d\left(\gamma H - \sigma_r\right)}{\gamma H - \sigma_r} + 2\frac{dr}{r} = 0\,. \tag{4.17}$$

Let us solve this equation:

$$\int \frac{d\left(\gamma H - \sigma_r\right)}{\gamma H - \sigma_r} = -2\int \frac{dr}{r}\,,$$

$$\ln\left|\gamma H - \sigma_r\right| = -2\ln\left|r\right| + \ln\left|C_1\right|,$$

$$\ln\left|\gamma H - \sigma_r\right| = \ln\left|\frac{C_1}{r^2}\right|,$$

$$\sigma_r = \gamma H - \frac{C}{r^2}\,, \tag{4.18}$$

where C is a unknown constant of integration. Define it from the boundary conditions

$$\sigma_r = \gamma H \ \text{ at } r \to \infty,$$
$$\sigma_r = p_0 \ \text{ at } r \to 1\,. \tag{4.19}$$

The first boundary condition is met automatically, and from the second one we obtain

$$C = \gamma H - p_0\,. \tag{4.20}$$

Substituting (4.20) into (4.18) and using (4.16) we obtain an equation for the corresponding expressions

$$\sigma_r = \gamma H - \frac{\gamma H - p_0}{r^2},$$
$$\sigma_\theta = \gamma H + \frac{\gamma H - p_0}{r^2} \tag{4.21}$$

or

$$\sigma_r = \gamma H\left(1 - \frac{1 - \frac{p_0}{\gamma H}}{r^2}\right),$$
$$\sigma_\theta = \gamma H\left(1 + \frac{1 - \frac{p_0}{\gamma H}}{r^2}\right). \tag{4.22}$$

Stress distribution around the excavation is shown at fig. 4.2.

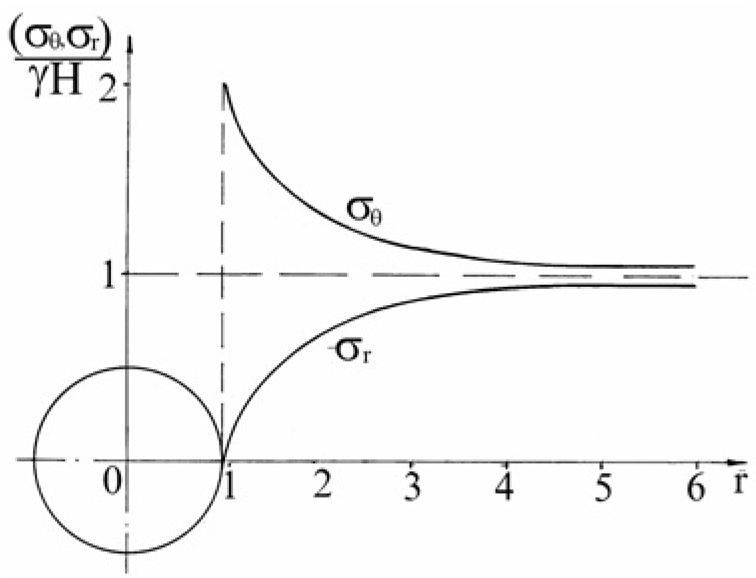

Fig. 4.2. Stress distribution around excavation in the rock mass

5. FIRST ORDER LINEAR DIFFERENTIAL. BERNOULLI'S EQUATION

Definition. A type of the equation

$$y' + P(x)y = Q(x). \tag{5.1}$$

is called a linear differential equation of the first order.

It is linear with respect to the unknown function y and its derivative y'. $P(x)$ and $Q(x)$ are given continuous functions of the argument x in some interval. For example the equation

$$y' + \frac{x}{1-x^2} \cdot y = \frac{ax}{1-x^2}, \quad a = const,$$

is a linear differential equation of the first order. Here

$$P(x) = \frac{x}{1-x^2}, \quad Q(x) = \frac{ax}{1-x^2}.$$

If the function $Q(x) \equiv 0$ then the equation (5.1) takes the form

$$\frac{dy}{dx} + P(x)y = 0 \tag{5.2}$$

and is called a linear homogeneous equation. The equation (5.1) in which $Q(x) \neq 0$ called a linear inhomogeneous.

Consider two ways to solve the differential equation (5.1).

The method of variation of arbitrary constant (Lagrangian method). The method consists in the following. First we find a general integral of the corresponding homogeneous equation with separable variables (5.2). Separating the variables and integrating, we have

$$\frac{dy}{y} = -P(x)dx, \quad y \neq 0,$$

$$\int \frac{dy}{y} = -\int P(x)dx + C_1, \quad \ln|y| = -\int P(x)dx + C_1,$$

$$|y| = e^{C_1 - \int P(x)dx}, \quad y = Ce^{-\int P(x)dx} \quad \left(C = \pm e^{C_1}\right).$$

We have obtained that a general solution of the homogeneous equation (5.2) has the form

$$y = Ce^{-\int P(x)dx}, \tag{5.3}$$

where C is arbitrary constant.

We will find a solution of the inhomogeneous equation (5.1) in the form (5.3). Let C is not a constant but a function of the argument x:

$$y = C(x)e^{-\int P(x)dx}, \tag{5.4}$$

where C is unknown function. To determine this function we find y' from (5.4) and substitute y and y' into equation (5.1). We obtain

$$C'(x)e^{-\int P(x)dx}-C(x)P(x)e^{-\int P(x)dx}+P(x)C(x)e^{-\int P(x)dx}=Q(x),$$
$$C'(x)e^{-\int P(x)dx}=Q(x),$$
$$C'(x)=Q(x)e^{\int P(x)dx}. \qquad (5.5)$$

Hence
$$C(x)=\int Q(x)e^{\int P(x)dx}dx+C^*, \qquad (5.6)$$

where C^* is arbitrary constant. Substituting (5.6) in (5.4) we find

$$y=\left(\int Q(x)e^{\int P(x)dx}dx+C^*\right)e^{-\int P(x)dx}. \qquad (5.7)$$

Since (5.6) is a general solution of the equation (5.5) then (5.7) is a general solution of the equation (5.1).

Method of substitution (Bernoulli's method). We seek a solution of equation (5.1) as a product of two functions $u(x)$ and $v(x)$:

$$y(x)=u(x)v(x). \qquad (5.8)$$

Hence $y'=u'v+uv'$. We substitute y and y' into the equation (5.1). Then we have
$$u'v+uv'+P(x)uv=Q(x)$$
or
$$u'v+u\left(v'+P(x)v\right)=Q(x). \qquad (5.9)$$

Let us choose the function v in such way that the expression in brackets would be equal to zero, i.e.

$$v'+P(x)v=0.$$

This is an equation with separable variables. We have:

$$\frac{dv}{dx}=-P(x)v, \quad \frac{dv}{v}=-P(x)dx,$$
$$\int\frac{dv}{v}=-\int P(x)dx, \quad \ln v=-\int P(x)dx,$$
$$v=e^{-\int P(x)dx} \qquad (5.10)$$

(a general solution is $v = Ce^{-\int P(x)dx}$; we take a particular solution at $C = 1$). This value v is substituted into (5.9):

$$u'e^{-\int P(x)dx} = Q(x), \quad u' = Q(x)e^{\int P(x)dx},$$

$$u = \int Q(x)e^{\int P(x)dx}dx + C. \tag{5.11}$$

Hence, we have found the expression for the functions v and u ((5.10) и (5.11)). Since $y = uv$ then the general solution is

$$\begin{gathered} y = e^{-\int P(x)dx}\left(C + \int Q(x)\varphi(x)dx\right), \\ \varphi(x) = e^{\int P(x)dx}. \end{gathered} \tag{5.12}$$

Bernoulli's equation. First-order equation of the form

$$y' + P(x)y = Q(x)y^{\alpha}, \quad \alpha \neq 0, \quad \alpha \neq 1, \tag{5.13}$$

where $P(x)$ and $Q(x)$ are given continuous functions is called the Bernoulli's equation. The equation (5.13) is a linear inhomogeneous equation $y' + P(x)y = Q(x)$ at $\alpha = 0$ and it is an equation with separable variables $y' + (P(x) - Q(x))y = 0$ at $\alpha = 1$. Therefore we assume that $\alpha \neq 0$, $\alpha \neq 1$.

Bernoulli's equation can be reduced to a linear equation by replacing the variable

$$z = y^{1-\alpha}, \quad y \neq 0.$$

We have:

$$y = z^{\frac{1}{1-\alpha}}, \quad y' = \frac{1}{1-\alpha}z^{\frac{1}{1-\alpha}} \cdot z'.$$

Substituting y' and y into (5.13) we obtain

$$\frac{1}{1-\alpha}z^{\frac{1}{1-\alpha}} \cdot z' + P(x)z^{\frac{1}{1-\alpha}} = Q(x)z^{\frac{\alpha}{1-\alpha}}.$$

Hence

$$z' + (1-\alpha)P(x)z = (1-\alpha)Q(x). \tag{5.14}$$

This is a linear equation relatively to function z. For the obtaining the general solution $z(x)$ of the linear equation (5.14) we use the formula (5.12). Now we find the general integral of the Bernoulli's equation

$$
\begin{gathered}
y^{1-\alpha}(x)=e^{(\alpha-1)\int P(x)dx}\left(C+(1-\alpha)\int Q(x)\varphi(x)dx\right), \\
\varphi(x)=e^{(1-\alpha)\int P(x)dx}.
\end{gathered} \tag{5.15}
$$

Note that at $\alpha>0$ the Bernoulli equation has the obvious solution $y\equiv 0$. The Bernoulli's equation can be solved using the method of substituting $y=uv$ where one of the functions selected by arbitrarily (Bernoulli's Method).

Example 5.1. Find the general integral of the equation

$$y'-2ye^x=2\sqrt{y}e^x.$$

Solution. This is Bernoulli's equation (see (5.13)). Here

$$P(x)=-2e^x, \quad Q(x)=2e^x, \quad \alpha=\frac{1}{2}.$$

We use the formula (5.15):

$$\int P(x)dx=-2\int e^x dx=-2e^x, \quad \varphi(x)=e^{\frac{1}{2}\left(-2e^x\right)}=e^{-e^x},$$

$$\int Q(x)\varphi(x)dx=2\int e^x\cdot e^{-e^x}dx=\left\{-e^x=t, e^x dx=-dt\right\}=-2\int e^t dt=-2e^{-e^x}.$$

The formula (5.15) takes the form

$$\sqrt{y}=e^{-\frac{1}{2}\left(-2e^x\right)}\left(C+\frac{1}{2}\left(-2e^{-e^x}\right)\right)=e^{e^x}\left(C-e^{-e^x}\right)=Ce^{e^x}-1, \quad C>0.$$

Answer: $\sqrt{y}=Ce^{e^x}-1$, $y=0$.

6. FIRST ORDER HOMOGENEOUS DIFFERENTIAL EQUATIONS

Definition. A function $f(x,y)$ is called a homogeneous of dimension m if for any x, y and $\lambda>0$ performed an identity

$$f(\lambda x,\lambda y)=\lambda^m f(x,y)$$

takes place.

Consider some examples.

Example 6.1.

$f(x,y)=5x^2-8xy$;

$f(\lambda x,\lambda y)=5(\lambda x)^2-8\lambda x\cdot\lambda y=\lambda^2(5x^2-8xy)$, then $f(\lambda x,\lambda y)=\lambda^2 f(x,y)$, $m=2$, that is a homogeneous function of dimension 2.

Example 6.2.

$f(x,y)=2x-3y+\sqrt{x^2+y^2}$;

$f(\lambda x,\lambda y)=2\lambda x-3\lambda y+\lambda\sqrt{x^2+y^2}=\lambda f(x,y)$, $f(x,y)$ is a homogeneous function of dimension 1.

Example 6.3.

$f_1(x,y)=\dfrac{x+y}{x-5y}$, $f_2(x,y)=\dfrac{x^3-3x^2y+4y^3}{2y^2x-3x^3}$, $f_3(x,y)=\dfrac{4y-5x}{\sqrt{3x^2-2y^2+xy}}$ are homogeneous functions of zero dimension.

Example 6.4.

$f(x,y)=\dfrac{\sin(x+y)}{\sin(x-y)}$ – this function is not homogeneous.

If $f(x,y)$ is a function of zero dimension then it can always be represented in the form $f(x,y)=F\left(\dfrac{y}{x}\right)$. For the example 6.3 we have

$$f_1(x,y)=\frac{1+\frac{y}{x}}{1-5\frac{y}{x}}=F_1\left(\frac{y}{x}\right),\ f_2(x,y)=\frac{1-3\frac{y}{x}+4\left(\frac{y}{x}\right)^3}{2\left(\frac{y}{x}\right)^2-3}=F_2\left(\frac{y}{x}\right),$$

$$f_3(x,y)=\frac{4\frac{y}{x}-5}{\sqrt{3-2\left(\frac{y}{x}\right)^2+\frac{y}{x}}}=F_3\left(\frac{y}{x}\right).$$

Definition 6.1. An equation

$$M(x,y)dx+N(x,y)dy=0 \tag{6.1}$$

called homogeneous if M and N are homogeneous functions of the same dimension.

Homogeneous equation (6.1) which is solved relatively the derivative y' has the form

$$y'=f\left(\frac{y}{x}\right). \tag{6.2}$$

To solve the equation (6.2) we use the substitution

$$\frac{y}{x}=t \quad (t \text{ is a function of the variable } x).$$

Hence $y=xt$, $y'=t+xt'$ $\left(t'=\dfrac{dt}{dx}\right)$. Then the equation (6.2) takes the form

$$t+xt'=f(t)$$

or

$$\frac{dt}{dx}=\frac{f(t)-t}{x}.$$

This is an equation with separable variables. When

$$f(t)-t\neq 0 \tag{6.3}$$

we share variables and integrate

$$\frac{dt}{f(t)-t}=\frac{dx}{x},$$

$$\int\frac{dt}{f(t)-t}=\ln|x|+C. \tag{6.4}$$

Substituting the values $t=\dfrac{y}{x}$ we obtain the general integral of equation (6.2) under condition (6.3). If the equation $f(t)-t=0$ has a root $t=t_0$ that apart from the general integral (6.4) the differential equation (6.2) has another solution $y=t_0x$.

Example 6.5. Find the general integral of the equation

$$xy'=y+\sqrt{y^2-x^2}. \tag{6.5}$$

Solution. We write this equation in the form

$$y'=\frac{y}{x}+\sqrt{\left(\frac{y}{x}\right)^2-1}. \tag{6.6}$$

This is a homogeneous differential equation. In (6.6) we perform a substitution

$$\frac{y}{x}=t, \quad y=xt, \quad y'=t+xt',$$

then

$$t + xt' = t + \sqrt{t^2 - 1}, \quad xt' = \sqrt{t^2 - 1}, \quad \frac{dt}{dx} = \frac{\sqrt{t^2 - 1}}{x}.$$

Separating the variables, we obtain

$$\frac{dt}{\sqrt{t^2 - 1}} = \frac{dx}{x}.$$

Let us integrate expression:

$$\int \frac{dt}{\sqrt{t^2 - 1}} = \int \frac{dx}{x} + C_1,$$

$$\ln\left|t + \sqrt{t^2 - 1}\right| = \ln|x| + \ln|C_2|, \quad C_2 \neq 0,$$

$$\left|t + \sqrt{t^2 - 1}\right| = |C_2 x|,$$

$$t + \sqrt{t^2 - 1} = \pm C_2 x,$$

$$t + \sqrt{t^2 - 1} = C_3 x, \quad C_3 \neq 0.$$

Now we substitute here $t = \dfrac{y}{x}$:

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1} = C_3 x, \quad C_3 \neq 0. \qquad (6.7)$$

Separating the variables we assumed that $x\left(t^2 - 1\right) \neq 0$. Let $x\left(t^2 - 1\right) = 0$. Then $x = 0$ or $t = \pm 1$. But $x = 0$ does not satisfy the equation (6.5). The solution $y = x$ corresponds to the root $t = 1$. The solution $y = -x$ corresponds to the root $t = -1$. The solutions $y = x$, $y = -x$ can not be obtained from the total solution (6.7) for any value of the arbitrary constant C_3. Therefore, the equation (6.5) has a general integral

$$\frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1} = Cx, \quad C \neq 0; \quad y = x; \quad y = -x.$$

7. HIGHER ORDER DIFFERENTIAL EQUATIONS. REDUCING THE ORDER OF EQUATIONS

One of the methods for solving the higher-order differential equation is a method of order reducing. Its essence is that using a suitable change of variable this differential equation reduces to a lower order equation.

Consider the types of such equations.

I. An equation of the form

$$y^{(n)} = f(x). \tag{7.1}$$

Knowing that $y^{(n)} = \left(y^{(n-1)}\right)'$ this equation is solved by successive integration n times:

$$y^{(n-1)} = \int f(x)\,dx + C_1,$$
$$y^{(n-2)} = \int\left(\int f(x)\,dx + C_1\right)dx + C_2,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

The last step: $y = \int y'(x)\,dx + C_n$.

II. Second-order equation that does not contain an unknown function. This is an equation of the form

$$f\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \tag{7.2}$$

Let $\dfrac{dy}{dx} = p$ then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$. We obtain the equation

$$f\left(x, p, \frac{dp}{dx}\right) = 0, \tag{7.3}$$

which is a first-order equation relatively the variable p. Let the general solution of equation (11.3) has the form $p = \varphi(x, C_1)$. We have $\dfrac{dy}{dx} = p = \varphi(x, C_1)$ whence

$$y = \int \varphi(x, C_1)\,dx + C_2.$$

III. Second-order equation which does not contain an independent variable. This is an equation of the form

$$f\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \tag{7.4}$$

Let $\dfrac{dy}{dx} = p$ and assume that p is a function of y. Then

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p.$$

Substituting

$$\frac{dy}{dx} = p, \quad \frac{d^2y}{dx^2} = \frac{dp}{dy} \cdot p$$

into the equation (7.4), we obtain the equation of the first order with respect to p

$$f\left(y, p, p\frac{dp}{dy}\right) = 0.$$

Let $p = \varphi(y, C_1)$ is a general solution of this equation. By integration of an equation with separable variables $\frac{dy}{dx} = \varphi(y, C_1)$ we find the general integral of the equation (7.4).

Example 7.1. Find the general solution of the equation

$$yy'' = (y')^2.$$

Solution. This is an equation of the form (7.4). Let $y' = p(y)$ then $y'' = p\frac{dp}{dy}$. We obtain an equation for p:

$$yp\frac{dp}{dy} = p^2.$$

This is an equation with separable variables. We have

$$\frac{dp}{p} = \frac{dy}{y}, \quad \int \frac{dp}{p} = \int \frac{dy}{y} + C_1,$$

$$\ln|p| = \ln|y| + \ln C_2, \quad p = C_3 y.$$

Here C_3 is an integration constant. If $C_3 = 0$ we obtain a solution $p = 0$ that is $y' = 0$ which satisfies the original equation. We have

$$\int \frac{dy}{y} = \int C_3 x dx + C_4, \quad \ln|y| = C_3 x + C_4,$$

$$|y| = e^{C_3 x + C_4}, \quad y = \pm e^{C_4} \cdot e^{C_3 x}.$$

Thus a function

$$y = C_1 e^{C_2 x}$$

is the general solution of the original equation where C_1, C_2 are the constants.

8. ONE-DIMENSIONAL MOTION. LAW OF ENERGY CONSERVATION

Consider the following problem of mechanics. Suppose that a body moves along the axis Ox under the action of the force which depends only on the coordinate $\boldsymbol{x}$. Newton's second law gives the differential equation of motion:

$$m\frac{d^2x}{dt^2} = F\left(x\right). \tag{8.1}$$

Let the motion has such initial conditions:

$$\begin{cases} t = 0, x = x_0, \\ \dfrac{dx}{dt} = v_0. \end{cases} \tag{8.2}$$

Differential equation (8.1) is the second-order equation in which there is no independent variable t. Using the contents of the Chapter 7 paragraph III we reduce the order of the equation. In our case, the unknown function is indicated by the letter x and the argument is indicated by the letter t. We have

$$\frac{dx}{dt} = v \tag{8.3}$$

(the derivative is denoted here by the letter v because it is a velocity). Then we can write

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}. \tag{8.4}$$

Substituting (8.3) and (8.4) into (8.1) we obtain the first order differential equation

$$mv\frac{dv}{dx} = F\left(x\right).$$

This is an equation with separable variables and we have consistently:

$$mvdv = F\left(x\right)dx,$$

$$\int mvdv = \int F\left(x\right)dx.$$

Let us replace indefinite integrals on definite integrals with variable upper limits. We choose these limits guided by the initial conditions (8.2):

$$\int_{v_0}^{v} mvdv = \int_{x_0}^{x} F(x)\,dx + C .$$

At $t=0$ we obtain from (8.2) $v=v_0$, $x=x_0$, that is

$$0=0+C,$$

whence $C=0$. Finally we obtain

$$\frac{mv^2}{2}-\frac{mv_0^2}{2}=\int_{x_0}^{x} F(x)\,dx . \tag{8.5}$$

The value $\frac{mv^2}{2}$ is a kinetic energy of a body.

Here the force F depends only on x. We introduce a characterization of the force field called a potential energy

$$u(x)=-\int_{a}^{x} F(x)\,dx ,$$

where a is an arbitrary point $a\in\mathbb{R}$. Physically, this means that when the force F performs a work

$$A=\int_{a}^{x} F(x)\,dx$$

then a potential energy decreases by the same value. Now we have from (8.5):

$$\int_{x_0}^{x} F(x)\,dx=\int_{a}^{x} F(x)\,dx-\int_{a}^{x_0} F(x)\,dx=-u(x)+u(x_0),$$

that is

$$\frac{mv^2}{2}-\frac{mv_0^2}{2}=-u(x)+u(x_0).$$

Hence we obtain

$$\frac{mv^2}{2}+u(x)=\frac{mv_0^2}{2}+u(x_0) \tag{8.6}$$

that is a law of the energy conservation. It means that the sum of kinetic and potential energy does not change with time. This is the simplest form of the most important laws of nature.

The equation (8.5) called "a first integral" of the equation (8.1). This means that the equation (8.1) is not solved but gives a connection between an unknown function x and its first derivative $v=\frac{dx}{dt}$. This gives the valuable information concerning the behavior of a body.

9. LINEAR n-ORDER EQUATIONS

Consider the basic concepts and Cauchy's problem for linear n-order equations.

Definition 9.1. Linear n-order differential equation has the form

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=f(x). \tag{9.1}$$

This equation is linear respectively to function y and its derivatives. Here $f(x)$, $a_1(x),\ldots,$ $a_n(x)$ are functions defined on the interval (a,b). Functions $a_i(x)$, $i=\overline{1,n}$ are called the coefficients of the equation (9.1) and a function $f(x)$ is the right-hand side of the equation or a free member.

Definition 9.2. If $f(x)\equiv 0$ then equation

$$y^{(n)}+a_1(x)y^{(n-1)}+a_2(x)y^{(n-2)}+\ldots+a_n(x)y=0 \tag{9.2}$$

is called homogeneous linear n-order equation. An equation (9.1) where $f(x)\neq 0$ is called inhomogeneous linear n-order equation or the right-part equation.

Theorem 9.1 (the existence and uniqueness of the Cauchy’s problem of solutions in terms the linear equation (9.1)). Let functions $a_i(x)$, $i=\overline{1,n}$, and $f(x)$ are continuous on the interval (a,b). Then for any $x_0\in(a,b)$ and for an arbitrary set of numbers $y_0,y_0',\ldots,y_0^{(n-1)}$ equation (9.1) has a unique solution considering the initial conditions

$$y(x_0)=y_0,\ y'(x_0)=y_0',\ \ldots,\ y^{(n-1)}(x_0)=y_0^{(n-1)}$$

on the interval (a,b).

Consequence 9.1. Let $y(x)$ be a solution of a linear homogeneous differential equation (9.2) which satisfies initial conditions

$$y(x_0)=0,\ y'(x_0)=0,\ \ldots,\ y^{(n-1)}(x_0)=0.$$

Then $y(x)\equiv 0\ \ \forall x\in(a,b)$.

Proof of the Theorem 9.1. Equation (9.1) can be written in the form:

$$y^{(n)}=f(x)-a_1(x)y^{(n-1)}-\ldots-a_n(x)y \tag{9.3}$$

or $y^{(n)} = F\left(x, y, y', \ldots, y^{(n-1)}\right)$ where the right side of equation (9.3) is denoted by F.

Partial derivatives of the function F with respect to variables $y, y', \ldots, y^{(n-1)}$ looks like

$$\frac{\partial F}{\partial y} = -a_n(x),\ \frac{\partial F}{\partial y'} = -a_{n-1}(x),\ \ldots,\ \frac{\partial F}{\partial y^{(n-1)}} = -a_1(x).$$

They are continuous on the interval (a,b). Consequently, equation (9.1) satisfies conditions of the Cauchy's theorem (Theorem 2.2) on the whole interval of continuity of functions $a_i(x)$, $i = \overline{1,n}$, $f(x)$. The theorem is proved.

We introduce the notion of a linear differential operator.

Definition 9.3. Let us denote the left-hand side of equation (9.1) as $L(y)$:

$$L(y) \equiv y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \ldots + a_n(x) y. \tag{9.4}$$

This designation is called as a linear differential operator of the function y.

The operator $L(y)$ has the following properties:

$$L(Cy) = CL(y),\ C = const; \tag{9.5}$$

$$L(y_1 + y_2) = L(y_1) + L(y_2). \tag{9.6}$$

Proof. According to (9.4) we have

$$\begin{gathered} L(Cy) = (Cy)^{(n)} + a_1(x)(Cy)^{(n-1)} + \ldots + a_n(x)Cy = \\ = Cy^{(n)} + a_1(x)Cy^{(n-1)} + \ldots + a_n(x)Cy = \\ = C\left(y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y\right) = CL(y). \end{gathered}$$

Formula (9.5) is proved. Similarly

$$\begin{gathered} L(y_1 + y_2) = (y_1 + y_2)^{(n)} + a_1(x)(y_1 + y_2)^{(n-1)} + \ldots + a_n(x)(y_1 + y_2) = \\ = y_1^{(n)} + y_2^{(n)} + a_1(x)\left(y_1^{(n-1)} + y_2^{(n-1)}\right) + \ldots + a_n(x)(y_1 + y_2) = \\ = \left(y_1^{(n)} + a_1(x)y_1^{(n-1)} + \ldots + a_n(x)y_1\right) + \left(y_2^{(n)} + a_1(x)y_2^{(n-1)} + \ldots + a_n(x)y_2\right) = \\ = L(y_1) + L(y_2). \end{gathered}$$

Formula (9.6) is proved.

On the basis of the properties of (9.5) and (9.6) we have

$$L(C_1y_1 + C_2y_2 + \ldots + C_ny_n) = C_1L(y_1) + C_2L(y_2) + \ldots + C_nL(y_n), \tag{9.7}$$

where $C_1,\ldots,C_n$ are constants.

Consider the properties of the linear equation solution.

Theorem 9.2. If functions $y_1(x),\ldots,y_n(x)$ are solutions of linear homogeneous differential equation (9.2) then their linear combination

$$C_1y_1 + C_2y_2 + \ldots + C_ny_n$$

is a solution of equation (9.2) as well. Here $C_1,\ldots C_n$ are constants.

Proof. Consider the equation (9.2) in the form $L(y)=0$. Since $y_1,\ldots,y_n$ are solutions of (13.2) then $L(y_1)=0$, $L(y_2)=0$, …, $L(y_n)=0$. Using (13.7) we obtain

$$L(C_1y_1 + C_2y_2 + \ldots + C_ny_n) = 0.$$

This means that the function $C_1y_1 + C_2y_2 + \ldots + C_ny_n$ satisfies the equation (9.2).

Theorem 9.3. If $y_1(x)$ and $y_2(x)$ are solutions of the inhomogeneous equation (9.1) then their difference $y_2(x)-y_1(x)$ is a solution of the homogeneous equation (9.2).

Proof. According to conditions of the theorem we have $L(y_1)=f(x)$, $L(y_2)=f(x)$. Using (9.7) we obtain

$$L(y_1-y_2)=L(y_1)-L(y_2)=f(x)-f(x)=0$$

that proves our assertion.

Theorem 9.4. If a complex function $y(x)=u(x)+iv(x)$ is a solution of the homogeneous equation (9.2) then its real part $u(x)$ and a coefficient of the imaginary part $v(x)$ are solutions of this equation.

Proof. We write the equality (9.2) in the form $L(y)=0$. Putting $y_1=u$, $y_2=v$, $C_1=1$, $C_2=i$ in the equality (9.7) we obtain

$$0 \equiv L(y)=L(u+iv)=L(u)+iL(v)$$

whence $L(u)\equiv 0$, $L(v)\equiv 0$.

Theorem 9.5 (a property of a superposition or imposing of solutions). Let an equation be given:

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = f_1(x) + \ldots + f_k(x). \tag{9.8}$$

If functions $y_i(x)$, $i=\overline{1,k}$ are solutions of the equations

$$y^{(n)}+a_1(x)y^{(n-1)}+\ldots+a_n(x)y=f_i(x),\ i=\overline{1,k}, \qquad (9.9)$$

then the function

$$y(x)=y_1(x)+y_2(x)+\ldots+y_k(x)$$

is a solution of the equation (9.8).

Proof. According to the theorem conditions $L(y_1)=f_1(x),\ldots,L(y_k)=f_k(x)$. Putting in the equation (9.7) $C_1=\ldots=C_k=1$ we obtain

$$L(y_1+\ldots+)y_k=L(y_1)+\ldots+L(y_k)=f_1(x)+\ldots+f_k(x).$$

This is a statement of the theorem.

10. LINEAR DEPENDENCE AND INDEPENDENCE OF FUNCTIONS

Definition 10.1. Let functions $y_1,y_2,\ldots,y_n$ defined on the interval (a,b) are connected by expression

$$\alpha_1 y_1+\alpha_2 y_2+\ldots+\alpha_n y_n=0, \qquad (10.1)$$

where $\alpha_1,\ldots,\alpha_n$ are some constants among which no one is equal to zero. Then these functions are linearly dependent in the interval (a,b).

If for a given system of functions the equality (10.1) is true only when $\alpha_1=\alpha_2=\ldots=\alpha_n=0$ then these functions are called linearly independent in the interval (a,b).

A simpler definition of linear dependence and independence can be given for two functions.

Definition 10.2. Functions $y_1(x)$ and $y_2(x)$ are linearly independent in the interval (a,b) if

$$\frac{y_1(x)}{y_2(x)}\neq const.$$

$$\frac{y_1(x)}{y_2(x)}=const,$$

then functions $y_1(x)$ and $y_2(x)$ are linearly dependent.

Consider criteria for linear dependence and independence of a system of functions.

Consider a system of functions

$$y_1(x), y_2(x), \dots, y_n(x) \tag{10.2}$$

which are $(n-1)$-times differentiable. Construct a determinant of n-th order

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}. \tag{10.3}$$

This determinant is called the Wronskian for system of functions (10.2).

Theorem 10.1. Let functions $y_1, \dots, y_n$ be $(n-1)$ time differentiable in the interval (a,b) and let these functions be linearly dependent. Then their Wronskian is equal to zero.

Proof (for case $n=3$). Let functions $y_1(x), y_2(x), y_3(x)$ are linearly dependent in the interval (a,b). Then for any $x \in (a,b)$ the equality is true

$$\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0, \tag{10.4}$$

where none of coefficients $\alpha_1, \alpha_2, \alpha_3$ equals to zero. For example, suppose $\alpha_3 \neq 0$. Then from (10.4) we obtain

$$y_3 = \beta_1 y_1 + \beta_2 y_2, \quad \beta_i = -\frac{\alpha_i}{\alpha_3}, \quad i = 1,2. \tag{10.5}$$

Differentiating (10.5) we have

$$y_3' = \beta_1 y_1' + \beta_2 y_2', \quad y_3'' = \beta_1 y_1'' + \beta_2 y_2''. \tag{10.6}$$

Let us construct Wronskian for the system of functions using (10.5) and (10.6):

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \beta_1 y_1 + \beta_2 y_2 \\ y_1' & y_2' & \beta_1 y_1' + \beta_2 y_2' \\ y_1'' & y_2'' & \beta_1 y_1'' + \beta_2 y_2'' \end{vmatrix}.$$

So the third column of the Wronskian is a linear combination of the first two columns. This determinant is equal to zero. The theorem is proved.

The sufficient condition for the linear independence of functions implies from the Theorem 10.1.

Theorem 10.1. If the Wronskian $W(x)$ of a system of functions $y_1,\dots,y_n$ is not equal to zero at least at one point x_0 of the interval (a,b), $W(x_0)\neq 0$, then the functions are linearly independent in (a,b).

Example 10.1. A system of functions

$$1,x,x^2,\dots,x^{n-1} \tag{10.7}$$

is linearly independent in the interval $(-\infty,+\infty)$.

We prove this for the case $n=4$ i.e. we show that the system of functions

$$1,x,x^2,x^3 \tag{10.8}$$

is linearly independent on the interval $(-\infty,+\infty)$.

Let us construct the Wronskian for the functions (10.8):

$$W(x)=\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix}.$$

Since $W(x)=1\cdot 1\cdot 2\cdot 6=12\neq 0\ \ \forall x\in(-\infty,+\infty)$ then according to the Theorem 10.2 system of functions (10.8) is linearly independent on $(-\infty,+\infty)$.

Example 10.2. If $k_1,k_2,\dots,k_n$ are different numbers then the system of functions

$$e^{k_1x},e^{k_2x},\dots,e^{k_nx} \tag{10.9}$$

is linearly independent in the interval $(-\infty,+\infty)$.

We prove this for the case $n=3$ i.e. we show that the functions

$$e^{k_1x},e^{k_2x},e^{k_3x} \tag{10.10}$$

are linearly independent in $(-\infty,+\infty)$ where k_1,k_2,k_3 are different numbers.

Wronskian for the functions (10.10) looks like

$$W(x)=\begin{vmatrix} e^{k_1x} & e^{k_2x} & e^{k_3x} \\ k_1e^{k_1x} & k_2e^{k_2x} & k_3e^{k_3x} \\ k_1^2e^{k_1x} & k_2^2e^{k_2x} & k_3^2e^{k_3x} \end{vmatrix}=e^{k_1x}\cdot e^{k_2x}\cdot e^{k_3x}\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{vmatrix};$$

$$\begin{vmatrix} 1 & 1 & 1 \\ k_1 & k_2 & k_3 \\ k_1^2 & k_2^2 & k_3^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ k_1 - k_3 & k_2 - k_3 & k_3 \\ k_1^2 - k_3^2 & k_2^2 - k_3^2 & k_3^2 \end{vmatrix} =$$

$$= (k_1 - k_3)(k_2 - k_3) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & k_3 \\ k_1 + k_3 & k_2 + k_3 & k_3^2 \end{vmatrix} = (k_1 - k_3)(k_2 - k_3)(k_2 - k_1).$$

Thus

$$W(x) = e^{(k_1 + k_2 k_3)} \cdot (k_1 - k_3)(k_2 - k_3)(k_2 - k_1) \neq 0$$

since the function $e^{\alpha x} > 0$ and $k_1 \neq k_2 \neq k_3$. According to the theorem 10.2 the system of functions (10.10) is linearly independent in the interval $(-\infty, +\infty)$.

Example 10.3. Functions

$$e^{kx}, xe^{kx}, x^2 e^{kx}, \ldots, x^{n-1} e^{kx} \tag{10.11}$$

are linearly independent in the interval $(-\infty, +\infty)$. Since $e^{kx} \neq 0$ and

$$C_1 e^{kx} + C_2 x e^{kx} + \ldots + C_n x^{n-1} e^{kx} = e^{kx} \left(C_1 + C_2 x + \ldots + C_n x^{n-1} \right)$$

then the linear independence of these functions follows from the example 10.1.

11. GENERAL SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

Theorem 11.1 (necessary and sufficient conditions for the linear independence of a homogeneous linear equation solution)**.** Let $y_1, \ldots y_n$ are solutions of a linear homogeneous differential equation (9.2) in the interval (a, b). Then

I. If the Wronskian of these solutions is $W(x) \neq 0$ at least at one point $x_0 \in (a, b)$ then the solutions $y_1(x), \ldots, y_n(x)$ are linearly independent in (a, b).

II. If the solutions $y_1(x), \ldots, y_n(x)$ of the equation (9.2) are linearly independent in (a, b) then their Wronskian $W(x) \neq 0 \ \forall x \in (a, b)$.

Consider the structure of the general solution of a homogeneous linear differential n-order equation.

Definition 11.1. A system of n linearly independent particular solutions $y_1(x), \ldots, y_n(x)$ of the equation (9.2) is called a fundamental system of solutions.

Theorem 11.2 (concerning the structure of equation (9.2) general solution). Fundamental system of solutions $y_1(x), \ldots, y_n(x)$ forms a basis of solutions dimension. A general solution of this equation looks like

$$y(x) = C_1 y_1(x) + \ldots + C_n y_n(x), \tag{11.1}$$

where $C_1,\ldots,C_n$ are constants.

Proof. The sum (11.1) is a solution of (9.2) at any $C_1,\ldots,C_n$ (Theorem 10.1). Let $\tilde{y}(x)$ be an arbitrary solution of this equation. We prove that there is only one set of constants $\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_n$ for $\tilde{y}(x)$ such that

$$\tilde{y}(x)=\tilde{C}_1 y_1(x)+\ldots+\tilde{C}_n y_n(x). \tag{11.2}$$

This means that a fundamental system of solutions forms a basis of a solutions dimension for the equation (9.2).

Let

$$\tilde{y}(x_0)=\tilde{y}_0,\ \tilde{y}'(x_0)=\tilde{y}'_0,\ldots,\ \tilde{y}^{(n-1)}(x_0)=\tilde{y}_0^{(n-1)}, \tag{11.3}$$

and $x_0\in(a,b)$. Let us consider the system of n linear algebraic equations relatively unknown values $C_1,\ldots,C_n$:

$$\begin{cases} C_1y_1(x_0)+C_2y_2(x_0)+\ldots+C_ny_n(x_0)=\tilde{y}_0, \\ C_1y_1'(x_0)+C_2y_2'(x_0)+\ldots+C_ny_n'(x_0)=\tilde{y}'_0, \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ C_1y_1^{(n-1)}(x_0)+C_2y_2^{(n-1)}(x_0)+\ldots+C_ny_n^{(n-1)}(x_0)=\tilde{y}_0^{(n-1)}. \end{cases} \tag{11.4}$$

A determinant of this system is the Wronskian related to the system of functions $y_1(x),\ldots,y_n(x)$ at the point x_0, that is $W(x_0)$. The determinant $W(x_0)\neq 0$ because the functions $y_1,\ldots,y_n$ are linearly independent solutions of the differential equation (9.2) according to Theorem 11.1. This means that the system (11.4) has a unique solution $\tilde{C}_1,\tilde{C}_2,\ldots,\tilde{C}_n$. Substituting the values $\tilde{C}_1,\ldots,\tilde{C}_n$ in (11.1) we obtain the solution of the equation in the form

$$y(x)=\tilde{C}_1y_1(x)+\tilde{C}_2y_2(x)+\ldots+\tilde{C}_ny_n(x). \tag{11.5}$$

Differentiating (11.5) $(n-1)$ times, we find

$$\begin{gathered} y'(x)=\tilde{C}_1y_1'(x)+\ldots+\tilde{C}_ny_n'(x), \\ y''(x)=\tilde{C}_1y_1''(x)+\ldots+\tilde{C}_ny_n''(x), \\ \ldots \\ y^{(n-1)}(x)=\tilde{C}_1y_1^{(n-1)}(x)+\ldots+\tilde{C}_ny_n^{(n-1)}(x). \end{gathered}$$

Let us substitute the value x_0 into the obtained equations and (11.5). Then, considering (11.4) we can conclude that the solutions $y(x)$ and $\tilde{y}(x)$ satisfies the same initial conditions:

$$y(x_0)=\tilde{y}_0,\, y'(x_0)=\tilde{y}'_0,\ldots,\, y^{(n-1)}(x_0)=\tilde{y}_0^{(n-1)}.$$

Considering the Theorem 9.1 we obtain the identity $\tilde{y}(x)\equiv y(x)$ that is the equality (9.2). Hence, the first assertion of the theorem is proved. This means that formula (11.1) contains some solution of the differential equation (9.2) i.e. the formula (15.1) defines a general solution of equation (9.2).

Let us examine the structure of a general solution of the inhomogeneous linear differential equation.

Theorem 11.3. The general solution of the inhomogeneous linear differential equation (9.1) $y_{о.н.}$ is equal to the sum of any particular solution $\overline{y}$ and the general solution $y_{о.о.}$ of the corresponding homogeneous equation (9.2):

$$y_{о.н.}=\overline{y}+y_{о.о.}. \tag{11.6}$$

Proof. Let $y_1(x),\ldots,y_n(x)$ be a solution fundamental system of the homogeneous equation (9.2). Let $y(x)$ is an arbitrary solution of the inhomogeneous equation (9.1) and $\overline{y}(x)$ is a particular solution of this equation. Consider the difference $y(x)-\overline{y}(x)$. According to the Theorem 9.4 this difference satisfies the equation (9.2), and hence, considering the Theorem 11.2 we obtain

$$y(x)-\overline{y}(x)=\sum_{i=1}^{n}\tilde{C}_i y_i(x),$$

where $\tilde{C}_i$ are constants (see the equality (11.2)). Hence for an arbitrary solution $y(x)$ of the inhomogeneous equation (9.1) we have

$$y(x)=\overline{y}(x)+\sum_{i=1}^{n}\tilde{C}_i y_i(x). \tag{11.7}$$

Thus a general solution of the inhomogeneous equation (13.1) has the form

$$y_{о.н.}=\overline{y}(x)+\sum_{i=1}^{n}\tilde{C}_i y_i(x).$$

According to Theorem 11.2 the second member is the general solution of the homogeneous equation (9.2). Equality (11.6) is proved.

12. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS

A linear homogeneous second order differential equation with constant coefficients has the form

$$y'' + a_1 y' + a_2 y = 0, \tag{12.1}$$

where a_1, a_2 are real numbers. According to the Theorem 11.2 a general solution of this equation looks like

$$y = C_1 y_1 + C_2 y_2, \tag{12.2}$$

where y_1, y_2 are two linearly independent particular solutions and C_1, C_2 are constants.

Let us find a solution of the equation (12.1) in the form

$$y = e^{\lambda x}, \tag{12.3}$$

where λ is a some constant (real or complex). The derivatives look like

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}.$$

Substituting them and the function (12.3) into the left-hand side of the equation (12.1) we obtain

$$e^{\lambda x}\left(\lambda^2 + a_1\lambda + a_2\right) = 0.$$

The multiplier $e^{\lambda x} \neq 0$ therefore

$$\lambda^2 + a_1\lambda + a_2 = 0. \tag{12.4}$$

Definition 12.1. Quadratic equation (12.4) with the unknown value λ is called the characteristic equation for the differential equation (12.1).

We know that three cases are possible concerning the roots of the equation (12.4): I) roots are real and distinct; II) roots are real and equal; III) complex roots. We consider each case separately.

I. If roots λ_1, λ_2 of the characteristic equation (12.4) are real and distinct that is $\lambda_1 \neq \lambda_2$ then particular solutions of the equation (12.1) are the functions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}.$$

These solutions are linearly independent:

$$\frac{y_1(x)}{y_2(x)} = e^{(\lambda_1 - \lambda_2)x} \not\equiv const,$$

so they form a fundamental system of solutions of the equation (12.1). A general solution of this equation according to (12.2) has the form

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{12.5}$$

II. Let the roots of the characteristic equation (12.4) be real and equal, that is $\lambda_1 = \lambda_2 = \lambda$. In this case the discriminant of the equation (12.4) is $a_1^2 - 4a_2 = 0$ and $\lambda = -\frac{a_1}{2}$. Therefore the differential equation (12.1) has a solution

$$y_1 = e^{\lambda x}. \tag{12.6}$$

We can show that the function $y_2 = xe^{\lambda x}$ is a solution of the equation (12.1) as well. The solution $y_2(x)$ is linearly independent with respect to (12.6) because

$$\frac{y_2(x)}{y_1(x)} = x \not\equiv const.$$

Therefore if the characteristic equation (12.4) has equal roots $\lambda_1 = \lambda_2 = \lambda$ then a general solution of the equation (12.1) has the form

$$y = C_1 e^{\lambda x} + xC_2 e^{\lambda x}. \tag{12.7}$$

III. Suppose that the characteristic equation (12.4) has complex roots. This case tares place if the discriminant $a_1^2 - 4a_2 < 0$. Here we have

$$\lambda_1 = -\frac{a_1}{2} + i\sqrt{a_2 - \frac{a_1^2}{4}}, \quad \lambda_1 = -\frac{a_1}{2} - i\sqrt{a_2 - \frac{a_1^2}{4}}.$$

Let us introduce the notation

$$\alpha = -\frac{a_1}{2}, \quad \beta = \sqrt{a_2 - \frac{a_1^2}{4}}.$$

Then

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta.$$

Thus, the roots of the characteristic equation (12.4) are complex conjugates. Substituting the values λ_1 and λ_2 into the formula (12.3) we obtain such complex solutions of the differential equation (12.1):

$$\tilde{y}_1 = e^{(\alpha+i\beta)x}, \quad \tilde{y}_2 = e^{(\alpha-i\beta)x}.$$

Let us use the Euler's formulas

$$e^{i\beta x} = \cos\beta x + i\sin\beta x, \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

Then partial solutions $\tilde{y}_1$ and $\tilde{y}_2$ of the equation (12.1) can be written as

$$\tilde{y}_1 = e^{\alpha x}\left(\cos\beta x + i\sin\beta x\right),$$
$$\tilde{y}_2 = e^{\alpha x}\left(\cos\beta x - i\sin\beta x\right).$$

Denote the real part of these differences as $y_1(x)$ and a coefficient of the imaginary part as $y_2(x)$. We have

$$\tilde{y}_1 = y_1(x) + iy_2(x), \quad \tilde{y}_2 = y_1(x) - iy_2(x),$$
$$y_1(x) = e^{\alpha x}\cos\beta x, \quad y_2(x) = e^{\alpha x}\sin\beta x.$$

On the basis of the Theorem 9.4 the functions $y = y_1(x)$ and $y = y_2(x)$ are the solutions of the differential equation (12.1). These solutions are linearly independent because

$$\frac{y_2(x)}{y_1(x)} = tg\beta x \not\equiv const.$$

So, in the case of complex roots $\lambda_{1,2} = \alpha \pm i\beta$ of the characteristic equation (12.4) a general solution of the differential equation (12.1) has the form

$$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x. \tag{12.8}$$

Example 12.1. Find a general solution of the equation

$$y'' - 3y' + 2y = 0. \tag{12.9}$$

Solution. We write the corresponding characteristic equation

$$\lambda^2 - 3\lambda + 2 = 0.$$

The roots of this equation $\lambda_1 = 1, \lambda_2 = 2$ are real and different. Therefore a general solution of the equation (12.9) has the form (12.5):

$$y = C_1 e^x + C_2 e^{2x}.$$

Example 12.2. Find a solution of the differential equation

$$y'' - 4y' + 29y = 0 \tag{12.10}$$

satisfying the initial conditions

$$y(0) = 1, \quad y'(0) = 7. \tag{12.11}$$

Solution. We write the characteristic equation

$$\lambda^2 - 4\lambda + 29 = 0.$$

The values

$$\lambda_1 = 2 + 5i, \quad \lambda_2 = 2 - 5i$$

are the roots of this equation. Then a general solution of the equation (12.10) has the form (12.8), where $\alpha = 2, \beta = 5$:

$$y = e^{2x}\left(C_1 \cos 5x + C_2 \sin 5x\right). \tag{12.12}$$

Now we use the initial conditions (12.11). For this we find

$$y' = 2e^{2x}\left(C_1 \cos 5x + C_2 \sin 5x\right) + e^{2x}\left(-5C_1 \sin 5x + 5C_2 \cos 5x\right).$$

To search for constants C_1 and C_2 we have a system of the equations:

$$\begin{cases} C_1 = 1, \\ 2C_1 + 5C_2 = 7. \end{cases}$$

Hence $C_1 = 1, C_2 = 1$. Substituting the constants $C_1 = 1, C_2 = 1$ in the (12.12) we find a solution

$$y = e^{2x} \cos 5x + e^{2x} \sin 5x$$

which satisfies the initial conditions (12.11).

Example 12.3. A body having a mass m moves along the Ox-axis under the force F which is proportional to the body x-position and directed unto the origin of coordinates. Find the law of the body motion if at the time $t = 0$ the x-coordinate is equal to x_0 and the body speed is equal to v_0.

Solution. Let us denote the body coordinate at the time t as $x(t)$. Then $x'(t)$ is the body speed at time t and its acceleration is $x''(t)$. Based on Newton's second law we can write

$$mx'' = F.$$

According to the problem condition $F = -\gamma x(t), \gamma > 0$. Thus the function $x(t)$ satisfies the equation

$$mx'' = -\gamma x.$$

We denote $\frac{\gamma}{m}$ as k^2 and write this equation in the form

$$x'' + k^2 x = 0. \tag{12.13}$$

This is a linear homogeneous second-order differential equation with constant coefficients. Let us find all solutions of this equation. We form a characteristic equation $\lambda^2 + k^2 = 0$. Hence $\lambda_{1,2} = \pm ik$. The function

$$x(t) = C_1 \sin kt + C_2 \cos kt$$

is a general solution of the equation (12.13), where C_1, C_2 are constants. We select from this set of solutions the solution that satisfies the initial condition

$$x(0) = x_0, \quad x'(0) = v_0. \tag{12.14}$$

Substituting the initial values $t = 0, x = x_0, x' = v_0$ into equalities

$$x(t) = C_1 \sin kt + C_2 \cos kt,$$
$$x'(t) = kC_1 \cos kt - kC_2 \sin kt.$$

We obtain an algebraic system of equations relatively C_1 and C_2:

$$\begin{cases} x_0 = C_2, \\ v_0 = kC_1; \end{cases} \Rightarrow \begin{cases} C_2 = x_0, \\ C_1 = \dfrac{v_0}{k}. \end{cases}$$

So a solution of the equation (12.13) which satisfies the conditions (12.14) is the function

$$x(t) = \frac{v_0}{k} \sin kt + x_0 \cos kt.$$

Let us represent this function as

$$x(t) = A_0\left(\frac{v_0}{kA_0}\sin kt + \frac{x_0}{A_0}\cos kt\right) = A_0\sin(kt+\theta_0), \qquad (12.15)$$

where

$$A_0 = \sqrt{\left(\frac{v_0}{k}\right)^2 + x_0^2}$$

and a value θ_0 is determined by equalities

$$\sin\theta_0 = \frac{x_0}{A_0}, \quad \cos\theta_0 = \frac{v_0}{kA_0}.$$

The equation (12.15) gives that any movement (except $x(t)\equiv 0$) described by the differential equation (12.13) is a simple harmonic oscillation with an amplitude A_0 and angular frequency k. A period of oscillation is $T = \frac{2\pi}{k}$ (this is a period of the function $\sin(kt+\theta_0)$).

Returning to the problem we can say that the body performs harmonic oscillations according to the law

$$x = A\sin\left(\sqrt{\frac{\gamma}{m}}\cdot t + \alpha\right),$$

where

$$A_0 = \sqrt{x_0^2 + \frac{m^2v_0^2}{\gamma^2}}, \quad \sin\alpha = \frac{x_0}{A}, \quad \cos\alpha = \frac{\sqrt{m}v_0}{\sqrt{\gamma}A}.$$

Here the value A is an amplitude and $T = 2\pi\sqrt{\frac{m}{\gamma}}$ is a period of these oscillations.

13. SOLUTION BY VARIATION OF THE CONSTANT

The following fact was established in Chapter 11. A solution of the inhomogeneous equation (9.1) can always be found if general solution of equation (9.2) is known and some particular solution of equation (9.1) is known as well. There is a general technics to find a particular solutions of the inhomogeneous equation (9.1) called the method of variation of arbitrary constants or Lagrange's method. We explain the essence of this method in terms of second order differential equation

$$y'' + a_1(x)y' + a_2(x)y = f(x). \tag{13.1}$$

Let a fundamental system of solutions is known for corresponding homogeneous equation

$$y'' + a_1(x)y' + a_2(x)y = 0. \tag{13.2}$$

Then

$$y = C_1 y_1(x) + C_2 y_2(x) \tag{13.3}$$

is a general solution of equation (13.2). Here C_1, C_2 are constants. We seek a particular solution $\overline{y}(x)$ of the equation (13.1) in the form (13.3) considering that values C_1, C_2 are not constants but functions of x:

$$y = C_1(x) y_1(x) + C_2(x) y_2(x). \tag{13.4}$$

The functions $C_1(x)$ and $C_2(x)$ can be chosen in such way that the function (13.4) will be a general solution of equation (13.1).

We obtain from the equality (13.4) by differentiating

$$y' = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2'. \tag{13.5}$$

Now we need to define two functions C_1 and C_2. One relation between them can be taken in arbitrary manner. We require that C_1 and C_2 satisfies the equality

$$C_1' y_1 + C_2' y_2 = 0. \tag{13.6}$$

Then (13.5) has the form

$$y' = C_1 y_1' + C_2 y_2'. \tag{13.7}$$

Differentiating this equality we obtain

$$y'' = C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2''. \tag{13.8}$$

We have:

$$\begin{gathered} C_1\left[y_1'' + a_1(x)y_1' + a_2(x)y_1\right] + \\ + C_2\left[y_2'' + a_1(x)y_2' + a_2(x)y_2\right] + \\ + C_1' y_1' + C_2' y_2' = f(x). \end{gathered} \tag{13.9}$$

Since y_1 and y_2 are the solutions of the homogeneous equation (13.2) then the expressions in square brackets of (13.9) are identically equal to zero. So we have the following relation:

$$C_1' y_1' + C_2' y_2' = f(x). \tag{13.10}$$

Combining the equalities (13.6) and (13.10) we obtain the system of equations

$$\begin{cases} C_1'y_1 + C_2'y_2 = 0, \\ C_1'y_1' + C_2'y_2' = f(x). \end{cases} \tag{13.11}$$

It is a system of two linear equations with two unknowns $C_1'(x)$ and $C_2'(x)$. This system has a unique solution since the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{13.12}$$

is the Wronskian $W(x) \neq 0 \ \ \forall x \in (a,b)$. Solving the system (13.11) we find

$$C_1'(x) = \varphi(x), \quad C_2'(x) = \psi(x),$$

where

$$\varphi(x) = -\frac{y_2(x) f(x)}{W(x)}, \quad \psi(x) = \frac{y_1(x) f(x)}{W(x)}. \tag{13.13}$$

Integrating, we obtain

$$C_1(x) = \int \varphi(x)dx + C_1^*, \quad C_2(x) = \int \psi(x)dx + C_2^*,$$

where C_1^*, C_2^* are arbitrary constants. Substituting the values of these functions in (13.4) we find a general solution of the inhomogeneous differential equation (13.1) in the form:

$$y = C_1^* y_1(x) + C_2^* y_2(x) + y_1(x)\int \varphi(x)dx + y_2(x)\int \psi(x)dx, \tag{13.14}$$

where the functions $\varphi(x), \psi(x)$ are defined by equalities (13.13), (13.12).

The first two summands gives a general solution of the equation (13.2) and the rest summands

$$y_1(x)\int \varphi(x)dx + y_2(x)\int \psi(x)dx$$

represents a particular solution of the inhomogeneous equation (13.1).

Example 13.1. Find a general solution of the equation

$$y'' - 3y' + 2y = \frac{e^{3x}}{1+e^{2x}}.$$

Solution. We obtain a fundamental system of solutions $y_1(x), y_2(x)$ of the corresponding homogeneous equation

$$y'' - 3y' + 2y = 0.$$

In the example 12.1 was have already obtained

$$y_1(x) = e^x, \quad y_2(x) = e^{2x}.$$

Wronskian of these solutions is

$$W(x) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}.$$

We find the functions $\varphi(x)$ and $\psi(x)$ ((13.13)), where $f(x) = \dfrac{e^{3x}}{1+e^{2x}}$:

$$\varphi(x) = -\frac{e^{2x}}{1+e^{2x}}, \quad \psi(x) = \frac{e^x}{1+e^{2x}}.$$

Hence

$$\int \varphi(x)\,dx = -\frac{1}{2}\ln\left(1+e^{2x}\right), \quad \int \psi(x)\,dx = arctg\, e^x.$$

Substituting $y_1(x), y_2(x), \int \varphi(x)dx, \int \psi(x)dx$ into (13.4) we obtain a general solution of the given equation

$$y(x) = C_1 e^x + C_2 e^{2x} - \frac{1}{2} e^x \ln\left(1+e^{2x}\right) + e^{2x} arctg\, e^x.$$

14. INHOMOGENEOUS LINEAR DIFFERENTIAL SECOND ORDER EQUATIONS WITH CONSTANT COEFFICIENTS

Let a_1 and a_2 be real numbers, $f(x)$ is given continuous function. An equation of the form

$$y'' + a_1 y' + a_2 y = f(x) \qquad (14.1)$$

is called a nonhomogeneous linear differential second order equation with constant coefficients.

The Theorem 11.3 asserts that a solution of this equation is a sum of one particular solution $\overline{y}(x)$ of the inhomogeneous equation (14.1) and a general solution $y_{o.o.}(x)$ of the corresponding homogeneous equation

$$y'' + a_1 y' + a_2 y = 0 . \tag{14.2}$$

The method of arbitrary constant variation described in Chapter 13 can be applied to equation (14.1) in any case. But sometimes it leads to a quadrature which is complex enough. To avoid this difficulties let us turn to so-called method of undetermined coefficients. This method allows finding the particular solution without quadrature at a certain structure of the equation (14.1) right-hand side.

We assume that the function $f(x)$ in the right-hand side of equation (14.1) has the form

$$f(x) = e^{\alpha x}\left(P_m(x)\cos\beta x + Q_n(x)\sin\beta x\right), \tag{14.3}$$

where $P_m(x)$ and $Q_n(x)$ are algebraic polynomials of degree m and n respectively, α and β are real numbers. Note that $P_m(x)$ and $Q_n(x)$ can be constant and one of them even equal to zero. Numbers α and β can be equal to zero as well.

A particular solution of equation (14.1) we seek in the form

$$\overline{y} = x^k e^{\alpha x}\left(R_j(x)\cos\beta x + S_j(x)\sin\beta x\right), \tag{14.4}$$

where $R_j(x)$ and $S_j(x)$ are algebraic polynomials with degree $j = \max\{m, n\}$; k is a multiplicity with which $\alpha \pm i\beta$ is one of the roots of the characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{14.5}$$

(if $\alpha \pm i\beta$ is not a root of the characteristic equation (14.5) then $k = 0$).

Example 14.1. Find a general solution of the equation

$$y'' + y' - 12y = e^{2x}(x-1). \tag{14.6}$$

Solution. We are looking for a general solution in the form

$$y = y_{o.o.} + \overline{y}, \tag{14.7}$$

where $y_{o.o.}(x)$ is a general solution of a corresponding homogeneous equation and $\overline{y}(x)$ is a particular solution of the given equation (14.6).

The corresponding characteristic equation looks like

$$\lambda^2 + \lambda - 12 = 0$$

whence $\lambda_1 = -4, \lambda_2 = 3$. So (formula (12.5))

$$y_{o.o.}=C_1e^{-4x}+C_2e^{3x}.$$

A right-hand side of equation (14.6) has the form (14.3). The number $\alpha=2$ is not a root of the characteristic equation. The polynomial $P_m(x)=x-1$ is a polynomial of first degree that is $m=1$. There are trigonometric functions so $\beta=0$. The particular solution of (14.6) can be written in the form (14.4) where $k=0$, $R_j(x)=Ax+B$, $\alpha=2$, $\beta=0$. We obtain:

$$\overline{y}=e^{2x}\left(Ax+B\right). \tag{14.8}$$

Let us find the derivatives

$$\overline{y}'=2e^{2x}\left(Ax+B\right)+Ae^{2x},$$
$$\overline{y}''=4e^{2x}\left(Ax+B\right)+4Ae^{2x}.$$

Substituting the function (14.8) and its derivatives into the equation (14.6) we obtain

$$-6e^{2x}\left(Ax+B\right)+5Ae^{2x}=e^{2x}\left(x-1\right)$$

whence

$$-6Ax-6B+5A=x-1.$$

Let us equate the coefficients at the same powers of x. So a system of equations is obtained

$$\begin{matrix} x^1: \\ x^0: \end{matrix}\begin{cases} -6A=1, \\ -6B+5A=-1. \end{cases}$$

Hence $A=-\dfrac{1}{6}, B=\dfrac{1}{36}$. Then a particular solution (14.8) equals

$$\overline{y}=e^{2x}\left(-\frac{1}{6}x+\frac{1}{36}\right)$$

and a general solution (14.7) can be written as

$$y=C_1e^{-4x}+C_2e^{3x}+e^{2x}\left(-\frac{1}{6}x+\frac{1}{36}\right).$$

Example 14.2. Find a general solution of the equation

$$y''-2y'+y=4x\sin x. \tag{14.9}$$

Solution. We write the characteristic equation

$$\lambda^2 - 2\lambda + 1 = 0.$$

The roots are $\lambda_1 = \lambda_2 = 1$. A general solution of the corresponding homogeneous equation has the form (12.7)

$$y_{o.o.} = e^x \left(C_1 + C_2 x\right). \qquad (14.10)$$

Right-hand side of the equation (14.9) has a form (14.3) where $\alpha = 0$, $\beta = 1$, $Q_n\left(x\right) = 4x$ that is $n = 1$, $P_m\left(x\right) = 0$. The number $\alpha + i\beta = i$ is not a root of the characteristic equation. A particular solution of the equation (14.9) we find in the form (14.4) where $k = 0$, $R_j\left(x\right)$ and $S_j\left(x\right)$ are polynomials of the first degree:

$$\overline{y} = \left(Ax + B\right)\cos x + \left(Cx + D\right)\sin x. \qquad (14.11)$$

We find derivatives

$$\begin{aligned}\overline{y}' &= A\cos x - \left(Ax + B\right)\sin x + C\sin x + \left(Cx + D\right)\cos x = \\ &= \left(A + Cx + D\right)\cos x + \left(C - Ax - B\right)\sin x, \\ \overline{y}'' &= C\cos x - \left(A + Cx + D\right)\sin x - A\sin x + \left(C - Ax - B\right)\cos x = \\ &= \left(2C - Ax - B\right)\cos x - \left(2A + Cx + D\right)\sin x.\end{aligned}$$

Substituting the function (14.11) and its derivatives into the equation (14.9) we find

$$\begin{aligned}&\left(2C - Ax - B\right)\cos x - \left(2A + Cx + D\right)\sin x - \left(2A + 2Cx + 2D\right)\cos x + \\ &+ \left(2C - 2Ax - 2B\right)\sin x + \left(Ax + B\right)\cos x + \left(Cx + D\right)\sin x = 4x\sin x.\end{aligned}$$

Equating the coefficients of the functions $\sin x$ and $\cos x$ in the left and right sides of the equality we obtain a system of equations

$$\begin{matrix}\cos x: \\ \sin x:\end{matrix}\begin{cases}C - A - Cx - D = 0, \\ -A - C + Ax + B = 2x.\end{cases}$$

Equate the coefficients at equal degrees of x in the first equation. Similarly akt in the second equation. We obtain a system of four equations with four unknowns

$$\begin{aligned} x^1 &: \\ x^0 &: \\ x^1 &: \\ x^0 &: \end{aligned}\begin{cases} -C = 0, \\ C - A - D = 0, \\ A = 2, \\ -A - C + B = 0. \end{cases}$$

Solving it we find $C = 0$, $A = 2$, $D = -2$, $B = 2$. Substituting these values A, B, C, D into the equality (14.11) one can obtain a particular solution

$$\overline{y} = (2x + 2)\cos x - 2\sin x .$$

A general solution of equation (14.9) $y = y_{o.o.} + \overline{y}$ has the form

$$y = e^x (C_1 + C_2 x) + (2x + 2)\cos x - 2\sin x .$$

15. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF n-TH ORDER WITH CONSTANT COEFFICIENTS

Definition 15.1. An equation of the form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0 , \tag{15.1}$$

where $a_1, \ldots, a_n$ are constants called homogeneous linear differential equation of n-th order with constant coefficients.

To solve this equation a fundamental system of solutions

$$y_1(x), y_2(x), \ldots, y_n(x).$$

must be found. Then by the Theorem 11.2 a function

$$y(x) = C_1 y_1 + C_2 y_2 + \ldots + C_n y_n \tag{15.2}$$

is a general solution of the equation (15.1). $C_1, \ldots, C_n$ are arbitrary constants.

Let us seek particular solutions in the form

$$y = e^{\lambda x} , \tag{15.3}$$

where λ is a constant. Then

$$y' = \lambda e^{\lambda x} , y'' = \lambda^2 e^{\lambda x} , \ldots, y^{(n)} = \lambda^n e^{\lambda x} .$$

Substituting the values of the derivatives and the function into (15.1) one can obtain

$$e^{\lambda x}\left(\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n\right) = 0.$$

Since $e^{\lambda x} \neq 0$ then

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0. \qquad (15.4)$$

Thus if λ is a root of the algebraic equation (15.4), then the function (15.3) is a solution of the equation (15.1), and vice versa.

Equation (15.4) is a characteristic equation corresponding to the differential equation (15.1) with constant coefficients

It is known that the equation (15.4) has *n* roots with multiplicities taken into account for each of them. Without going into details of the theory, we formulate necessary statements.

Theorem 15.1.

I. A particular solution $y = e^{\lambda x}$ of the differential equation (15.1) corresponds to each simple root λ of the equation (15.4).

II. Partial solutions

$$e^{\lambda x}, xe^{\lambda x}, \ldots, x^{m-1}e^{\lambda x}$$

corresponds to each real root λ of multiplicity $m > 1$.

III. Two particular solutions

$$e^{\alpha x}\sin\beta x, e^{\alpha x}\cos\beta x$$

corresponds to each pair $\alpha \pm \beta i$ of simple complex conjugate roots.

IV. To each pair $\alpha \pm \beta i$ of complex conjugate roots of multiplicity $m > 1$ $2m$ particular solutions of the form

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \ldots, x^{m-1}e^{\alpha x}\sin\beta x,$$
$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \ldots, x^{m-1}e^{\alpha x}\cos\beta x.$$

correspond.

The total sum of all root multiplicities related to equation (15.4) is equal to n. Therefore according to the Theorem 15.1 a number of partial solutions of the equation (15.1) is equal to n that is coincides with the order of the equation (15.1). We denote these partial solutions through $y_1, \ldots, y_n$. It can be shown that the obtained partial solutions are linearly independent and a general solution of equation (15.1) is given by the formula (15.2).

Example 15.1. The numbers $\lambda_1 = \lambda_2 = 4$, $\lambda_{3,4} = 2 \pm i$ are the roots of the characteristic equation. Write a general solution of a corresponding differential equation.

Solution. In accordance with the statement **II** of the Theorem 15.1 two particular solutions $y_1 = e^{4x}$, $y_2 = xe^{4x}$ corresponds to the roots $\lambda_{1,2} = 4$. In accordance with the statement **III** of the Theorem 15.1 two particular solutions

$y_3 = e^{2x}\cos x$, $y_4 = e^{2x}\sin x$ corresponds to the roots $\lambda_{3,4} = 2 \pm i$. A general solution of the equation is found according to the formula (15.2):

$$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 e^{2x}\cos x + C_4 e^{2x}\sin x .$$

Example 15.2. Find a general solution of the equation

$$y^{(5)} + 6y''' + 9y' = 0 .$$

Solution. The characteristic equation looks like

$$\lambda^5 + 6\lambda^3 + 9\lambda = 0 .$$

We have

$$\lambda^5 + 6\lambda^3 + 9\lambda = \lambda\left(\lambda^4 + 6\lambda^2 + 9\right) = \lambda\left(\lambda^2 + 3\right)^2 = 0 .$$

This implies that the characteristic equation has a simple real root $\lambda_1 = 0$ and complex conjugate roots $\lambda_{2,3} = \pm\sqrt{3}i$ of multiplicity 2. In accordance with the statements **I** and **IV** this differential equation has the linearly independent solutions:

$$y_1 = e^{0\cdot x} = 1,\ y_2 = \sin\sqrt{3}x,\ y_3 = x\sin\sqrt{3}x,$$
$$y_4 = \cos\sqrt{3}x,\ y_5 = x\sin\sqrt{3}x.$$

A general solution of this differential equation has the form

$$y(x) = C_1 + C_2\sin\sqrt{3}x + C_3 x\sin\sqrt{3}x + C_4\cos\sqrt{3}x + C_5 x\cos\sqrt{3}x .$$

16. SYSTEMS OF DIFFERENTIAL EQUATIONS IN NORMAL FORM

Definition 16.1. The system of differential equations

$$\frac{dx_i}{dt} = f_i\left(t, x_1, x_2, \ldots, x_n\right) \quad \left(i = \overline{1,n}\right) \tag{16.1}$$

is called a system in normal form. Here the derivatives of the unknown functions $x_i = x_i(t)$ are represented explicitly.

Definition 16.2. A set of continuously differentiable on I functions $x_i = \varphi_i(t)$, $i = \overline{1,n}$ such that for all $t \in I$

$$\frac{d\varphi_i}{dt} = f_i\left(t, \varphi_1(t), \ldots, \varphi_n(t)\right)$$

is called a solution of the system of equations (16.1) in the interval $I \subset \mathbb{R}$.

Consider one of the methods for solving the systems of differential equations in normal form. It consists in bringing the system of equations to a single n-th order equation or several equations of order less than n.

The scheme is as follows: we differentiate for example the first of the equations (16.1) successively $(n-1)$ time. We substitute instead derivatives $\frac{dx_i}{dt}$ their values. We have:

$$\begin{aligned}
&\frac{dx_1}{dt} = F_1\left(t, x_1, x_2, \ldots, x_n\right),\\
&\frac{d^2x_1}{dt^2} = F_2\left(t, x_1, x_2, \ldots, x_n\right),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n-1}x_1}{dt^{n-1}} = F_{n-1}\left(t, x_1, x_2, \ldots, x_n\right),\\
&\frac{d^n x_1}{dt^n} = F_n\left(t, x_1, x_2, \ldots, x_n\right).
\end{aligned} \tag{16.2}$$

We define $x_2, x_3, \ldots, x_n$ of the first $(n-1)$ equations of the system (16.2). Substituting these expressions in the last equation, we obtain a differential n-th order equation

$$\frac{d^n x_1}{dt^n} = F\left(t, x_1, \frac{dx_1}{dt}, \ldots, \frac{d^{n-1}x_1}{dt^{n-1}}\right).$$

Solving this equation we find a solution of the original system of equations.

Example 16.1. Solve the system of equations

$$\begin{cases} \dfrac{dx}{dt} = y^2 - \cos t, \\[2mm] \dfrac{dy}{dt} = \dfrac{x}{2y}. \end{cases}$$

Solution. We differentiate both sides of the first equation:

$$\frac{d^2x}{dt^2} = 2y\frac{dy}{dt} + \sin t. \tag{16.3}$$

From the second equation we have

$$2y\frac{dy}{dt}=x. \tag{16.4}$$

Substituting (16.4) into (16.3)one can obtain :

$$\frac{d^2x}{dt^2}=x+\sin t,$$
$$\frac{d^2x}{dt^2}-x=\sin t. \tag{16.5}$$

We received an inhomogeneous linear equation of the second order with constant coefficients. Its solution has the structure (11.6). We form the characteristic equation

$$\lambda^2-1=0.$$

It has two real distinct roots $\lambda_1=1$ and $\lambda_2=-1$. Hence

$$x_{o.o.}=C_1e^t+C_2e^{-t}.$$

Right-hand side of the equation (16.5) corresponds to the form (14.3). Therefore, a particular solution of equation (16.5) will be sought in the form (14.4), where $\alpha=0$, $\beta=1$, $\alpha\pm\beta i=\pm i$, $k=0$, R_j and S_j are first-degree polynomials:

$$\overline{x}=A\cos t+B\sin t.$$

We calculate the derivatives:

$$\overline{x}'=-A\sin t+B\cos t,$$
$$\overline{x}''=-A\cos t-B\sin t.$$

Substitute the values of the function and its derivatives in the equation (16.5):

$$-A\cos t-B\sin t-A\cos t-B\sin t=\sin t,$$
$$-2A\cos t-2B\sin t=\sin t.$$

Equate the coefficients of the trigonometric functions in the left and right sides of the equation:

$$\begin{array}{c|l} \cos t & -2A=0, \\ \sin t & -2B=1. \end{array}$$

Hence $A=0, B=-\frac{1}{2}$. Therefore,

$$\bar{x}=-\frac{1}{2}\sin t.$$

A general solution of the equation (16.5) has the form

$$x=C_1e^t+C_2e^{-t}-\frac{1}{2}\sin t. \qquad (16.6)$$

From the first equation of the original system we have

$$y^2=\frac{dx}{dt}+\cos t.$$

Differentiating (16.6) we find

$$\frac{dx}{dt}=C_1e^t-C_2e^{-t}-\frac{1}{2}\cos t.$$

Then

$$y^2=C_1e^t-C_2e^{-t}-\frac{1}{2}\cos t+\cos t,$$

$$y^2=C_1e^t-C_2e^{-t}+\frac{1}{2}\cos t.$$

QUESTIONS

1. What equation is called a differential equation?
2. How can we determine an order of a differential equation?
3. What is a general solution of a differential equation?
4. What is a particular solution of a differential equation?
5. Specify Cauchy's problem for a first order differential equation.
6. Give the mechanical illumination of Cauchy's problem for a second order differential equation.
7. Specify the differential equation with separable variables.
8. Specify the linear first order differential equation.
9. Specify Bernoulli's differential equation and display the solution techniques.
10. Give a definition of the first order homogeneous equation and specify the solution algorithm.
11. How can we reduce an order of a differential equation $f(x,y',y'')=0$?
12. How can we reduce an order of a differential equation $f(y,y',y'')=0$?
13. Specify an n-order linear homogeneous differential equation.
14. Specify an n-order linear inhomogeneous differential equation.

15. What features have the solutions of a linear differential equation?
16. Give a definition of function linear dependence and independence.
17. Write down the Wronskian.
18. Specify the general solution of a n-order linear homogeneous differential equation.
19. Specify the general solution of a n-order linear inhomogeneous differential equation.
20. How does a homogeneous second-order differential equation with constant coefficients look like?
21. How can we write a characteristic equation for a homogeneous second-order differential equation with constant coefficients?
22. Specify the general solution of a homogeneous second-order equation depending on the roots of the characteristic equation.
23. Explain the method of a constant variation.
24. Specify the function in a right-hand part of an inhomogeneous second-order differential equation for which we can select a particular solution.
25. Give a definition of a differential equation system in a normal form.
26. Specify a solution of a differential equation system.

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TABLE OF DERIVATIVES

1.	$C' = 0$	7a.	$\left(e^u\right)' = e^u \cdot u'$
2.	$x' = 1$	8.	$\left(\log_a u\right)' = \dfrac{1}{u}\log_a e \cdot u' \quad (a = const)$
3.	$\left(u + v - w\right)' = u' + v' - w'$	8a.	$\left(\ln u\right)' = \dfrac{1}{u} \cdot u'$
4.	$\left(uv\right)' = u'v + uv'$	9.	$\left(\sin u\right)' = \cos u \cdot u'$
4a.	$\left(Cv\right)' = Cv'$	10.	$\left(\cos u\right)' = -\sin u \cdot u'$
5.	$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$	11.	$\left(\operatorname{tg} u\right)' = \dfrac{1}{\cos^2 u} \cdot u' = \sec^2 u \cdot u'$
5a.	$\left(\dfrac{C}{v}\right)' = -\dfrac{Cv'}{v^2}$	12.	$\left(\operatorname{ctg} u\right)' = -\dfrac{1}{\sin^2 u} \cdot u' = -\operatorname{cosec}^2 u \cdot u'$
5b.	$\left(\dfrac{u}{C}\right)' = \dfrac{u'}{C}$	13.	$\left(\arcsin u\right)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u'$
6.	$\left(u^\alpha\right)' = \alpha u^{\alpha-1} \cdot u' \quad (\alpha = const)$	14.	$\left(\arccos u\right)' = -\dfrac{1}{\sqrt{1-u^2}} \cdot u'$
6a.	$\left(\sqrt{u}\right)' = \dfrac{1}{2\sqrt{u}} \cdot u'$	15.	$\left(\operatorname{arctg} u\right)' = \dfrac{1}{1+u^2} \cdot u'$
6b.	$\left(\dfrac{1}{u}\right)' = -\dfrac{1}{u^2} \cdot u'$	16.	$\left(\operatorname{arcctg} u\right)' = -\dfrac{1}{1+u^2} \cdot u'$
7.	$\left(a^u\right)' = a^u \ln a \cdot u' \quad (a = const)$	17.	$\left(u^v\right)' = vu^{v-1} \cdot u' + u^v \ln u \cdot v'$

Appendix 2

TABLE OF INTEGRALS

1.	$\int du = u + C$	9.	$\int \frac{du}{\cos^2 u} = \int \sec^2 u\, du = \operatorname{tg} u + C$
2.	$\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$	10.	$\int \frac{du}{\sin^2 u} = \int \operatorname{cosec}^2 u\, du = -\operatorname{ctg} u + C$
2a.	$\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$	11.	$\int \frac{du}{\sin u} = \ln\left\|\operatorname{tg}\frac{u}{2}\right\| + C$
2b.	$\int \frac{du}{u^2} = -\frac{1}{u} + C$	12.	$\int \frac{du}{\cos u} = \ln\left\|\operatorname{tg}\left(\frac{u}{2} + \frac{\pi}{4}\right)\right\| + C$
3.	$\int \frac{du}{u} = \ln\|u\| + C$	13.	$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\operatorname{arctg}\frac{u}{a} + C$
4.	$\int a^u du = \frac{a^u}{\ln a} + C$	13a.	$\int \frac{du}{u^2 + 1} = \operatorname{arctg} u + C$
4a.	$\int e^u du = e^u + C$	14.	$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left\|\frac{u-a}{u+a}\right\| + C$
5.	$\int \sin u du = -\cos u + C$	15.	$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
6.	$\int \cos u du = \sin u + C$	15a.	$\int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C$
7.	$\int \operatorname{tg} u du = -\ln\|\cos u\| + C$	16.	$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left\|u + \sqrt{u^2 \pm a^2}\right\| + C$
8.	$\int \operatorname{ctg} u du = \ln\|\sin u\| + C$	17.	$\int u dv = uv - \int v du$

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Навчальне видання

Сдвижкова Олена Олександрівна
Олевська Юлія Борисівна
Бабець Дмитро Володимирович
Коротка Лариса Іванівна

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